THE

UNIVERSITY OF MISSOURI

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MATHEMATICS SERIES

VOLUME I NUMBER 1

ON THE DEFINITION OF THE SUM OF A DIVERGENT SERIES

BY

LOUIS LAZARUS SILVERMAN, Ph.D.

Instructor in Mathematics in Cornell University

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CONTENTS

§ 1. INTRODUCTION *

The series $u_0 + u_1 + u_2 + \cdots$ is defined to be *convergent* whenever $\underset{n=\infty}{\mathrm{L}} (u_0 + u_1 + \cdots + u_n)$ exists; and the value of this limit is called the *sum* of the series. If this limit does not exist, the series is said to be *divergent.*

Some writers call a series divergent only when $\underset{n=\infty}{\mathrm{L}} (u_0+u_1+\cdots+u_n) = \infty$; all series which neither converge to a finite limit nor diverge to infinity are then called oscillatory.† The present considerations are limited to series which are oscillatory. We shall follow, however, the terminology of most writers‡ by calling divergent all series which do not converge; stating expressly, if necessary, when a series diverges to infinity.

A necessary condition for the convergence of a series is $\underset{n=\infty}{\mathrm{L}} u_n = 0$. Thus only a limited number of series can be dealt with. It is accordingly desirable to extend the definition of the sum of a series, so as to include a larger number of series with which we may deal rigorously. Our object will be to retain the class of convergent series, and to add to that set, by means of a more general definition, as large a class as possible of series which are not convergent. In order to be able to deal with these new series, however, we shall wish to preserve several fundamental properties of convergent series. We shall, in fact, demand the following fundamental requirements of any generalized definition of the sum of a series:

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* This paper was accepted as a dissertation by the Graduate Faculty of the University of Missouri in May, 1910, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

† Bromwich: *An Introduction to the Theory of Infinite Series*, p. 2.

‡ See e. g., Goursat-Hedrick: *Mathematical Analysis*, p. 327.

(i) The generalized sum must exist, whenever the series converges.

(ii) The generalized sum must be equal to the ordinary sum, whenever the series converges.

(iii) Each of the series

$$\begin{cases} u_0 + u_1 + u_2 + \cdots \\ \phantom{u_0 + {}} u_1 + u_2 + \cdots \end{cases}$$

has a generalized sum, whenever the other has, and $t = s - u_0$, if s and t are their respective sums.

(iv) If each of the series

$$\begin{cases} u_0 + u_1 + u_2 + \cdots \\ v_0 + v_1 + v_2 + \cdots \end{cases}$$

has a generalized sum, A and B respectively, then the series $(u_0 + v_0) + (u_1 + v_1) + (u_2 + v_2) + \cdots$ has a generalized sum which is $A + B$.

(v) If the series $u_0 + u_1 + u_2 + \cdots$ has s for its generalized sum, then $ku_0 + ku_1 + \cdots$ has a generalized sum which is ks.

I wish to express my gratitude to Professor E. R. Hedrick for his interest in my work, and to acknowledge my indebtedness to him for many helpful and important suggestions. I am also indebted to Drs. W. A. Hurwitz and H. M. Sheffer for many suggestions and criticisms.

§ 2. HISTORICAL RESUMÉ *

The earliest interest in divergent series centers about the series

$$1 - 1 + 1 - 1 + \cdots.$$

If we assume that this series has a generalized sum s, then the series, obtained by dropping the first term, $-1 + 1 - 1 + 1 \cdots$ must, by the third fundamental requirement of page 2, also have a generalized sum which is obviously $-s$. We have then, $s - 1 = -s$ or $s = \frac{1}{2}$. Thus, if the series is to have any value at all, that value must be $\frac{1}{2}$. And this is precisely the value which Leibniz† was led to attach to the series, by different considerations. The sum of n terms of the series is 0 or 1 according as n is even or odd; and since this sum is just as often equal to 1 as it is to 0, its probable value is the arithmetic mean, $\frac{1}{2}$. This same value was later attached to the series by Euler,‡ in a more satisfactory, though not entirely rigorous manner. "Let us say that the *sum* of any infinite series is the finite expression, by the expansion of which the series is generated. In this sense, the sum of the infinite series $1 - x + x^2 - x^3 \cdots$ will be $1/(1 + x)$, because the series arises from the expansion of the fraction, whatever number is put in place of x."§ In particular,

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + \cdots.$$

* The best historical sketches are to be found in Borel: *Leçons sur les Series Divergentes:* Introduction, and in an article by Pringsheim given immediately below.

† See Pringsheim: *Encyclopädie der Math. Wiss.*, I, 1, p. 107, note.

‡ *Instit. Calc. Diff.* (1755), Paris, II (p. 289).

§This quotation is taken from Bromwich, loc. cit., p. 266.

It is true, as has already been intimated, that none of the methods given above, to prove that the series should have the value $\frac{1}{2}$, is satisfactory from a theoretical point of view. But objections have been raised* to the result for practical reasons also. Thus, the series $1 - 1 + 1 - 1 + \cdots$ may be obtained from the expansion

$$\frac{1 + x}{1 + x + x^2} = \frac{1 - x^2}{1 - x^3} = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + \cdots$$

and setting $x = 1$,

$$\tfrac{2}{3} = 1 - 1 + 1 - 1 + \cdots.$$

To meet this difficulty, Lagrange† observed that we should write

$$\frac{1 + x}{1 + x + x^2} = 1 + 0 \cdot x - x^2 + x^3 + 0 \cdot x^4 - x^5 + \cdots,$$

so that for $x = 1$, we have

$$\tfrac{2}{3} = 1 + 0 - 1 + 1 + 0 - 1 + \cdots.$$

If we now follow the method of Leibniz, we see that the sequence corresponding to this series has, out of every three succeeding terms, once the value 0 and twice the value 1; its sum is accordingly $\frac{2}{3}$. Thus, Lagrange has removed the practical objection. Moreover the above method has been put on a rigorous theoretical foundation, by means of the following proposition,‡ which is a generalization of Abel's theorem:

THEOREM A:§ *If* $s_n = u_0 + u_1 + u_2 + \cdots + u_n$ *and*

$$\underset{n=\infty}{\mathrm{L}}\left[\frac{s_0 + s_1 + \cdots + s_n}{n + 1}\right] = s,$$

---

* By Callet. See reference immediately below.

† *Rapport sur le Memoire de Callet,* in: *Memoires de la classes des Sciences mathematiques et physiques de l'Institut,* t. III.

‡ Frobenius: *Journal de Crelle,* t. 89, p. 262.

§ Theorems embodying new results we shall indicate by numerals; all other theorems will be lettered A, B, C, $\cdots$.

then

$$\underset{x=1}{\mathrm{L}} \sum_{0}^{n} u_n x^n = s.$$

Thus, in the case of the series $1 - 1 + 1 - 1 + \cdots$,

$$\underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_n}{n} = \frac{1}{2},$$

and accordingly $\underset{x \doteq 1}{\mathrm{L}} (1 - x + x^2 \cdots) = \frac{1}{2}$; so that we may define the value of the series $1 - 1 + 1 \cdots$ to be $\underset{x \doteq 1}{\mathrm{L}} (1 - x + x^2 - x^3 + \cdots)$, or what amounts to the same thing,

$$\underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_n}{n},$$

whenever the limit exists.

The first mathematician actually to carry through the definition was Cesàro,* who approached the subject from another standpoint. Cauchy has defined as the product† of two series

$$\begin{cases} u_0 + u_1 + \cdots \\ v_0 + v_1 + \cdots \end{cases}$$

the series

$$u_0 v_0 + (u_0 v_1 + u_1 v_0) + (u_0 v_2 + u_1 v_1 + u_2 v_0) + \cdots;$$

this definition being justified by the theorem, due also to Cauchy, that the product series thus defined of two absolutely convergent series, is itself absolutely convergent. Mertens‡ has generalized this theorem by proving that the Cauchy product of an absolutely convergent series by a simply convergent series is convergent. The product of two simply convergent series may, however, be divergent. Cesàro has studied the divergent series which result from the product of two simply convergent series, and has obtained the following remarkable theorem:

* *Bulletin des Sciences mathematiqués*, t. XIV, 1890.

† We shall later refer to this as the Cauchy-product.

‡ *Journal de Crelle*, t. 79, p. 182.

THEOREM B: *Let the two series*

$$\begin{cases} u_0 + u_1 + u_2 + \cdots \\ v_0 + v_1 + v_2 + \cdots \end{cases}$$

converge to u and v respectively, and let

$$\begin{cases} w_n = (u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0) \\ s_n = w_0 + w_1 + \cdots + w_n \end{cases}$$

then

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n + 1} = u \cdot v.$$

The two theorems which we have stated justify us in stating the following definition:

Definition: If $s_n = u_0 + u_1 + u_2 + \cdots + u_n$, the series $u_0 + u_1 + \cdots + u_n + \cdots$ is summable and has the value s whenever*

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n + 1} = s.$$

Let us now proceed to show that this definition satisfies the fundamental requirements of page 2. To this end, we shall prove the following theorems.

THEOREM C:† *If a series converges, it is summable, and the two definitions give the same sum.*

Let $s_n = u_0 + u_1 + \cdots + u_n$, and $\mathop{\mathrm{L}}_{n=\infty} s_n = s$; we shall prove that

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n + 1} = s.$$

We have:

$$\left| \frac{s_0 + s_1 + \cdots + s_n}{n + 1} - s \right|$$
$$\equiv \left| \frac{(s_0 - s) + (s_1 - s) + \cdots + (s_q - s) + (s_{q+1} - s) + \cdots + (s_n - s)}{n + 1} \right|$$
$$\leq \frac{|s_0 - s| + |s_1 - s| + \cdots + |s_{q-1} - s|}{n + 1} + \frac{|s_q - s| + \cdots + |s_n - s|}{n + 1}.$$

* Cesàro calls series of this type *simply indeterminate*.

† By this theorem requirements (i) and (ii) are satisfied.

Since $\underset{n=\infty}{\mathbf{L}}\, s_n = s$, we can take q so great that $|s_i - s| < e/2$, $i \geq q$. Having chosen this q, let L be the largest of the numbers, $|s_i - s|$, $i = 0, 1, 2, \cdots q - 1$. Then we obtain:

$$\left|\frac{s_0 + s_1 + \cdots + s_n}{n+1} - s\right| \leq \frac{qL}{n+1} + \frac{(n-q+1)e}{2(n+1)} < \frac{qL}{n+1} + \frac{e}{2}$$

We can now choose n so large, $n > r$, that

$$\frac{qL}{n+1} < \frac{e}{2},$$

and hence,

$$\left|\frac{s_0 + s_1 + \cdots + s_n}{n+1} - s\right| < e, \quad n > r.$$

$$\therefore \underset{n=\infty}{\mathbf{L}}\left[\frac{s_0 + s_1 + \cdots + s_n}{n+1}\right] = s.$$

THEOREM D:* *Each of the series*

$$\begin{cases} u_0 + u_1 + u_2 + \cdots \\ \qquad u_1 + u_2 + \cdots \end{cases}$$

is summable when the other is; and s and t, their respective sums, are connected by the relation $s - u_0 = t$.

We shall prove only one part of this theorem, the method for the second part being exactly the same. We begin by proving the following fact.

Lemma: If the sequence $s_0, s_1, \cdots s_n, \cdots$ is summable and has s for its sum, then the sequence $s_1, s_2, \cdots s_n, \cdots$ is also summable, its sum being likewise s.

For,

$$\underset{n=\infty}{\mathbf{L}} \frac{s_1 + s_2 + \cdots + s_{n+1}}{n+1} = \underset{n=\infty}{\mathbf{L}} \frac{s_0}{n+1} + \underset{n=\infty}{\mathbf{L}} \frac{s_1 + \cdots + s_{n+1}}{n+1}$$

$$= \underset{n=\infty}{\mathbf{L}} \frac{s_0 + s_1 + \cdots + s_{n+1}}{n+1} = \underset{n=\infty}{\mathbf{L}} \frac{s_0 + s_1 + \cdots + s_{n+1}}{n+2} \cdot \frac{n+2}{n+1}$$

$$= \underset{n=\infty}{\mathbf{L}} \frac{s_0 + s_1 + \cdots + s_n}{n+1} = s.$$

* By this theorem requirement (iii) is satisfied.

To return now to Theorem D; we wish to prove that if $u_0 + u_1 + u_2 + \cdots$ is summable to s, then $u_1 + u_2 + \cdots$ is summable to $s - u_0$. The sequence corresponding to the series $u_0 + u_1 + u_2 + \cdots$ is u_0, $u_0 + u_1$, $\cdots$. By the lemma proved above, it follows that the sequence u_0+u_1, $u_0+u_1+u_2$, $\cdots$ or s_1, s_2, $\cdots$ is summable to s. The sequence corresponding to $u_1 + u_2 + \cdots$ is u_1, $u_1 + u_2$, $\cdots$ which may be written $s_1 - u_0$, $s_2 - u_0$, $\cdots$. Now

$$\underset{n=\infty}{\mathrm{L}}\left[\frac{(s_1 - u_0) + (s_2 - u_0) + \cdots + (s_n - u_0)}{n}\right] = \underset{n=\infty}{\mathrm{L}}\left(\frac{s_1 + s_2 + \cdots + s_n}{n} - u_0\right) = s - u_0.$$

Theorem E:* *If*

$$\begin{cases} u_0 + u_1 + \cdots \\ v_0 + v_1 + \cdots \end{cases}$$

are summable to u and v respectively, then the series $(u_0 + v_0) + (u_1 + v_1) + \cdots$ is summable to $u + v$.

Writing $s_n = u_0 + u_1 + \cdots + u_n$, $t_n = v_0 + v_1 + \cdots + v_n$, we have $s_n + t_n = (u_0 + v_0) + (u_1 + v_1) + \cdots + (u_n + v_n)$. We obtain:

$$\underset{n=\infty}{\mathrm{L}} \frac{(s_0 + t_0) + (s_1 + t_1) + \cdots + (s_n + t_n)}{n + 1} = \underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_n}{n + 1} + \underset{n=\infty}{\mathrm{L}} \frac{t_0 + t_1 + \cdots + t_n}{n + 1} = u + v.$$

Cesàro's definition of summability has accordingly been justified from the theoretical standpoint of our requirements for any generalized definition. We may naturally ask the practical question: how large is the class of series with which this definition enables us to deal? A partial answer to this question is contained in the following proposition:

* By this theorem requirement (iv) is satisfied. See also note p. 19.

THEOREM F: *A necessary condition for the summability of the series* $u_0 + u_1 + \cdots + u_n \cdots$ *is*

$$\underset{n=\infty}{\mathrm{L}} \frac{u_n}{n} = 0.$$

Since the series is summable,

$$\underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} - \underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_n}{n+1} = 0$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} - \underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_n}{n}$$

$$= - \underset{n=\infty}{\mathrm{L}} \frac{s_n}{n} = 0.$$

Hence:

$$\underset{n=\infty}{\mathrm{L}} \frac{u_n}{n} = \underset{n=\infty}{\mathrm{L}} \frac{s_n - s_{n-1}}{n} = \underset{n=\infty}{\mathrm{L}} \frac{s_n}{n} - \underset{n=\infty}{\mathrm{L}} \frac{s_{n-1}}{n} = 0.$$

We are accordingly limited to series for which

$$(1) \qquad \underset{n=\infty}{\mathrm{L}} \frac{u_n}{n} = 0.$$

But such a simple series as $1 - 2 + 3 - 4 + 5 \cdots$ fails to satisfy this condition. Furthermore, this series can be easily evaluated by following out the principle of Euler; for if we put $x = 1$ in the expansion:

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 \cdots,$$

we obtain

$$\tfrac{1}{4} = 1 - 2 + 3 - 4 + \cdots.$$

We are thus led to extend, with Cesàro, the above definition of summability of order 1, to summability of order 2. We say that a series is summable of order 2, if

$$\underset{n=\infty}{\mathrm{L}} \frac{(n+1)s_0 + ns_1 + \cdots + 2s_{n-1} + s_n}{\frac{(n+1)(n+2)}{2}} = s.$$

A necessary condition* for the existence of this limit is that

$$\underset{n=\infty}{\mathrm{L}} \frac{u_n}{n^2} = 0,$$

so that we cannot evaluate the series,

$$1 - r + \frac{r(r+1)}{2!} - \frac{r(r+1)(r+2)}{3!} + \cdots, \quad r > 2,$$

although we obtain by Euler's method,

$$\frac{1}{(1+x)^r} = 1 - rx + \frac{r(r+1)}{2!}x^2 - \frac{r(r+1)(r+2)}{3!}x^3 + \cdots$$

and accordingly

$$\frac{1}{2^r} = 1 - r + \frac{r(r+1)}{2!} - \frac{r(r+1)(r+2)}{3!} + \cdots.$$

We are thus led to state the following more general definition:

Definition:† The series $u_0 + u_1 + u_2 + \cdots$ is summable of order r, if r is the smallest integer for which there exists the limit:

$$(2) \quad \underset{n=\infty}{\mathrm{L}} \frac{s_0 \dfrac{r(r+1)\cdots(r+n-1)}{n!} + s_1 \dfrac{r(r+1)\cdots(r+n-2)}{(n-1)!} + \cdots + s_{n-2}\dfrac{r(r+1)}{2!} + s_{n-1}r + s_n}{\dfrac{(r+1)(r+2)\cdots(r+n)}{n!}}.$$

This definition includes convergence for $r = 0$; it also includes the other definitions given above for $r = 1, 2$ respectively. We shall not prove that this definition satisfies the requirements of page 2; this is easily verified.‡

Let us now return to Cesàro's first definition, and observe that we may generalize it in a more natural way.

* Bromwich, loc. cit., p. 318.

† Cesàro, loc. cit.

‡ This is done in a more general case, infra, pp. 55–57.

Definition:* Let

$$
(3)\quad \begin{cases} t_n^{(1)} = \dfrac{s_0 + s_1 + \cdots + s_n}{n + 1}, \\[2ex] t_n^{(r+1)} = \dfrac{t_0^{(r)} + t_1^{(r)} + \cdots + t_n^{(r)}}{n + 1}, \quad r = 1, 2, \cdots, \end{cases}
$$

then the smallest integer r for which $\underset{n=\infty}{\mathrm{L}}\, t_n^{(r)}$ exists, shall make the series summable of order r.

To distinguish this definition from that on page 10, we shall call the definitions Cesàro-summability of order r and Hölder-summability of order r, denoting them briefly by (C_r) and (H_r) respectively. It is known† that these two definitions are equivalent for the same r.

We may now ask how big a class of series this generalized definition enables us to deal with. If a series is (C_r), then‡

$$\underset{n=\infty}{\mathrm{L}} \frac{u_n}{n^r} = 0.$$

Accordingly the series $1 - t + t^2 - t^3 + \cdots$ $(t > 1)$ does not have a sum (C_r) for any value of r; since

$$\underset{n=\infty}{\mathrm{L}} \frac{t^n}{n^r} \neq 0, \quad t > 1.$$

We are thus led to generalize still further the definition for the sum of a series.

From the definition given on page 10, it is clear that we may write Cesàro's forms as follows:

$$s = \underset{n=\infty}{\mathrm{L}} \left[\frac{a_0 s_0 + a_1 s_1 + \cdots + a_n s_n}{a_0 + a_1 + \cdots + a_n} \right],$$

* Hölder: *Mathematische Annalen*, Bd. 20, p. 535.

† Schnee: *Math. Annalen*, Vol. LXVII (1909), p. 110.
Ford: *Am. Journal of Math.*, Vol. XXXII (1909), p. 315.

‡ Borel, *Series divergentes*, p. 92.

where the a_i are functions of both n and r, r being fixed.* Let us choose as our definition†

$$s = \underset{r=\infty}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}} \left[\frac{a_0(r)s_0 + a_1(r)s_1 + \cdots + a_n(r)s_n}{a_0(r) + a_1(r) + \cdots + a_n(r)} \right].$$

In particular we shall take $a_p(r) = r^p/p!$, and obtain

$$(4) \qquad s = \underset{r=\infty}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}} \left[\frac{s_0 + s_1\frac{r}{1} + s_2\frac{r^2}{2!} + \cdots + s_n\frac{r^n}{n!}}{1 + \frac{r}{1} + \frac{r^2}{2!} + \cdots + \frac{r^n}{n!}} \right] = \underset{r=\infty}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}}\ e^{-r} \left\{ s_0 + s_1\frac{r}{1} + \cdots + s_n\frac{r^n}{n!} \right\}.$$

It can be proved readily‡ that this limit exists, whenever the series converges. We shall now transform§ this limit.

Let||

$$\begin{cases} s(r) = s_0 + s_1\frac{r}{1} + s_2\frac{r^2}{2!} + \cdots + s_n\frac{r^n}{n!} + \cdots, \\ s'(r) = s_1 + s_2\frac{r}{1} + s_3\frac{r^2}{2!} + \cdots + s_{n+1}\frac{r^n}{n!} + \cdots, \end{cases}$$

then

$$u_1(r) = s'(r) - s(r) = u_1 + u_2\frac{r}{1} + u_2\frac{r^2}{2!} + \cdots + u_n\frac{r^n}{n!} + \cdots.$$

But

$$\frac{d}{dr}[e^{-r}s(r)] = e^{-r}[s'(r) - s(r)],$$

so that

$$e^{-r}s(r) = \int_0^r e^{-r}[s'(r) - s(r)]dr + u_0$$

and

$$s - u_0 = \int_0^\infty e^{-r}u_1(r)dr.$$

* Borel, *Series divergentes*, p. 94.

† r is now a positive real number.

‡ Bromwich, loc. cit., p. 298. This is a special case of Th. 12, p. 52 (infra).

§ Borel, loc. cit., p. 97.

|| It is assumed that $s(r)$ is convergent for all values of r; otherwise the limit (4) would have no meaning.

If now we integrate by parts we obtain:

$$s - u_0 = \left[e^{-r} \int_0^r u_1(r)dr \right]_0^\infty + \int_0^\infty e^{-r} \left[\int_0^r u_1(r)dr \right] dr,$$

or, if we let:

$$u(r) = u_0 + u_1 r + u_2 \frac{r^2}{2!} + \cdots + u_n \frac{r^n}{n!} + \cdots = u_0 + \int_0^r u_1(r)dr,$$

$$s - u_0 = [e^{-r}\{u(r) - u_0\}]_0^\infty + \int_0^\infty e^{-r}[u(r) - u_0]dr$$

$$= [e^{-r}u(r)]_0^\infty - u_0[e^{-r}]_0^\infty + \int_0^\infty e^{-r}u(r)dr - u_0 \int_0^\infty e^{-r}dr$$

$$= [e^{-r}u(r)]_0^\infty + \int_0^\infty e^{-r}u(r)dr,$$

i. e.,

$$s - u_0 = \operatorname*{L}_{r=\infty} [e^{-r}u(r)] - u_0 + \int_0^\infty e^{-r}u(r)dr,$$

or

$$s = \operatorname*{L}_{r=\infty} [e^{-r}u(r)] + \int_0^\infty e^{-r}u(r)dr.$$

If now we assume* that $\int_0^\infty e^{-r}u(r)dr$ is convergent, then it follows from the last equation that $\operatorname*{L}_{r=\infty} [e^{-r}u(r)]$ must exist. But this limit must necessarily be zero, for otherwise, the integral would not converge. Hence we obtain

$$(5) \qquad \left\{ \begin{aligned} & s = \int_0^\infty e^{-r}u(r)dr, \\ \text{where} \quad & u(r) = u_0 + u_1 \frac{r}{1} + u_2 \frac{r^2}{2!} + \cdots + u_n \frac{r^n}{n!} + \cdots, \end{aligned} \right.$$

whenever the integral converges. It can be proved† here, too,

* We have gone into greater detail here than does Borel, loc. cit., p. 98. But this is essentially his argument.

† Bromwich, loc. cit., p. 269.

that when the series $u_0 + u_1 + \cdots + u_n + \cdots$ converges, so does the above integral, and their values are the same.

Furthermore Borel proves the following theorem:

THEOREM G:* *If the Borel-integral definition*† *applies to the series:*

$$u_1 + u_2 + \cdots + u_n + \cdots = s,$$

then it also applies to the series $u_0 + u_1 + u_2 + \cdots$, *giving for its sum* $s + u_0$.

The converse, however, is not necessarily true. Thus if the series $u_0 + u_1 + u_2 + \cdots$ is summable by (5), it does not follow‡ that the series $u_1 + u_2 + \cdots$ is summable by (5). Since this fact is opposed to the requirement (iii), page 2, we are led to modify the above integral definition, and to state, with Borel, the following generalization:

Definition: The series $u_0 + u_1 + u_2 + \cdots$ *shall be called absolutely summable, whenever the integrals* $\int_0^\infty e^{-r} \mid u(r) \mid dr$, $\int_0^\infty e^{-r} \mid u^{(\lambda)}(r) \mid dr$ *converge, where* λ *denotes the order of any derivative.*

That this definition satisfies requirement (iii) is proved by the following theorem: §

THEOREM H: *If either of the series*

$$\begin{cases} u_0 + u_1 + u_2 + \cdots \\ \phantom{u_0 + {}} u_1 + u_2 + \cdots \end{cases}$$

is absolutely summable, so is the other; and if s, t *be their respective values, we have* $s - u_0 = t$.

We shall not enter into the further generalizations which have been given by Borel himself and by Le Roy.||

* Borel, loc. cit., p. 101.

† We shall call the two definitions given by Borel, the *Borel-mean* and the *Borel-integral* definition respectively.

‡ For an example, see Hardy, *Quarterly Journal*, Vol. 35 (1903), p. 30.

§ Borel, loc. cit.

|| Le Roy: *Annales de la Faculté de Sciences de Toulouse* (2° series), t. 2 (1902), p. 317. See p. 60, footnote.

§ 3. AVERAGEABLE SEQUENCES

On page 4 we have considered the series

$$\begin{cases} 1 - 1 + 1 - 1 + \cdots \\ 1 + 0 - 1 + 1 + 0 - 1 + \cdots, \end{cases}$$

and, replacing them by their respective sequences, we obtained

$$\begin{cases} \frac{1}{2} = 1,\ 0,\ 1,\ 0,\ \cdots \\ \frac{2}{3} = 1,\ 1,\ 0,\ 1,\ 1,\ 0,\ \cdots. \end{cases}$$

The probability-method of Leibniz* consists in taking for the sum of the sequence, the *average* of its limit-values. This method has been justified by the theorems of Frobenius† and Cesàro,‡ and the further generalizations. We propose now to give a justification of the method from another point of view.

To define the sum of a sequence as the average of its limit-values is obviously not adequate; for although we can tell that the limit 1 is to be counted twice in the sequence considered above,

$$1,\ 1,\ 0,\ 1,\ 1,\ 0,\ \cdots,$$

it is not easy or even possible to state the multiplicity of the limit-values in general, as is evident from the following example:

$$s_0,\ s_1,\ s_2,\ \cdots s_n,\ \cdots \quad \left.\begin{matrix} s_i = 0,\ i \neq n^2 \\ s_i = 1,\ i = n^2 \end{matrix}\right\} n = 0, 1, 2, \cdots.$$

To meet this difficulty, we shall proceed as follows.

Let us assume, to be concrete,§ that the sequence

$$s_0,\ s_1,\ s_2,\ \cdots s_n,\ \cdots$$

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* See page 3.

† See page 4.

‡ See page 5.

§ We shall go into every detail in only this simple case; the later generalizations we shall outline only briefly.

has two limit-values l_1 and l_2. Then we have

$$|s_m - l_1| < e, \quad |s_n - l_2| < e,$$

for an infinite number of values of m and of n, provided $m, n > N$. Having chosen e and N, let us now choose $i > N$; then there will be m of these i numbers s_i which fall in the interval about l_1, and n which fall in the interval about l_2. Since m and n are functions of i, we may write $m = f_1(i)$, $n = f_2(i)$. If we choose e sufficiently small, and $i > N$, we shall have

$$f_1(i) + f_2(i) + k = i,$$

where k is a constant independent of i.

Definition: The sequence $s_0, s_1, s_2, \cdots s_n, \cdots$, having l_1 and l_2 as limit-values, shall be called averageable and have s for its sum provided

$$\underset{i=\infty}{\mathrm{L}}\left[\frac{f_1(i)l_1 + f_2(i)l_2}{f_1(i) + f_2(i)}\right] = s.$$

That this limit, when it exists, does not depend upon the particular e we have chosen follows at once. For if we take $\bar{e} < e$, calling the corresponding functions $\bar{f}_1(i)$ and $\bar{f}_2(i)$, it is clear that

$$\left.\begin{aligned} f_1(i) &= \bar{f}_1(i) + k_1 \\ f_2(i) &= \bar{f}_2(i) + k_2 \end{aligned}\right\}$$

where k_1, k_2 are independent of i. We accordingly have:

$$\underset{i=\infty}{\mathrm{L}}\left[\frac{\bar{f}_1(i)l_1 + \bar{f}_2(i)l_2}{\bar{f}_1(i) + \bar{f}_2(i)}\right] = \underset{i=\infty}{\mathrm{L}}\left\{\frac{[f_1(i) - k_1]l_1 + [f_2(i) - k_2]l_2}{[f_1(i) - k_1] + [f_2(i) - k_2]}\right\}$$

$$= \underset{i=\infty}{\mathrm{L}}\left[\frac{\dfrac{f_1(i) - k_1}{f_1(i)}f_1(i)l_1 + \dfrac{f_2(i) - k_2}{f_2(i)}f_2(i)l_2}{\dfrac{f_1(i) - k_1}{f_1(i)}f_1(i) + \dfrac{f_2(i) - k_2}{f_2(i)}f_2(i)}\right]$$

$$= \underset{i=\infty}{\mathrm{L}}\left[\frac{f_1(i)l_1 + f_2(i)l_2}{f_1(i) + f_2(i)}\right],$$

since

$$\mathop{\mathrm{L}}_{i=\infty} \frac{k_1}{f_1(i)} = \mathop{\mathrm{L}}_{i=\infty} \frac{k_2}{f_2(i)} = 0.$$

Let us now find the sum of the sequence suggested on page 15,

$$1;\ 0,\ 0;\ 1,0,0,0,0;\ 1,0,0,0,0,0,0;\ \cdots,$$

i. e.,

$$\left.\begin{aligned} s_i &= 1, \quad i = n^2 \\ &= 0, \quad i \neq n^2 \end{aligned}\right\}.$$

Let us choose $i = m$, and let n^2 be the largest square integer less than or equal to m. Then we have:

$$s = \mathop{\mathrm{L}}_{m=\infty} \frac{n \cdot 1 + (m-n) \cdot 0}{m} = \mathop{\mathrm{L}}_{m=\infty} \frac{n}{m} = 0,$$

since $n^2 \leq m$.

Let us now see whether this definition satisfies the requirements of page 2. The first two requirements are obviously satisfied. As to the third, we observe that corresponding to the series $u_0 + u_1 + u_2 + \cdots + u_n + \cdots$; $u_1 + u_2 + \cdots + u_n + \cdots$, we have the sequences $s_0, s_1, s_2, \cdots s_n, \cdots$; $s_1 - u_0$, $s_2 - u_0$, $\cdots s_n - u_0, \cdots$; and if the limit-values of the first sequence, which will be assumed to be averageable to s, be l_1 and l_2, then those of the second sequence are $l_1 - u_0$, $l_2 - u_0$. We accordingly have:

$$\mathop{\mathrm{L}}_{i=\infty}\left[\frac{f_1(i)[l_1-u_0]+f_2(i)[l_2-u_0]}{f_1(i)+f_2(i)}\right] = \mathop{\mathrm{L}}_{i=\infty}\left[\frac{f_1(i)l_1+f_2(i)l_2}{f_1(i)+f_2(i)}\right] - u_0 = s - u_0.$$

We shall now show that the fourth requirement is satisfied.

THEOREM 1: *The sum of two averageable sequences is itself averageable, and has for its value the sum of their respective values.*

Let the two sequences

$$\begin{cases} s_0,\ s_1,\ s_2,\ \cdots\ s_n,\ \cdots \\ t_0,\ t_1,\ t_2,\ \cdots\ t_n,\ \cdots \end{cases}$$

have l_1, l_2 and m_1, m_2 as their respective limit-values, and s and t as their respective sums. Then we have:

$$s = \mathop{\mathbf{L}}_{i=\infty}\left[\frac{f_1(i)l_1 + f_2(i)l_2}{f_1(i) + f_2(i)}\right],$$

$$t = \mathop{\mathbf{L}}_{i=\infty}\left[\frac{g_1(i)m_1 + g_2(i)m_2}{g_1(i) + g_2(i)}\right].$$

We wish to show that the sequence

$$s_0 + t_0,\ s_1 + t_1,\ \cdots\ s_n + t_n,\ \cdots$$

is averageable, and has for its value $s + t$. We observe that the only limit-values for the sum-sequence are $l_1 + m_1$, $l_1 + m_2$, $l_2 + m_1$, $l_2 + m_2$. Let us call $F_{ij}(n)$ the number of the $(s_{\dot{n}} + t_{\dot{n}})$ which are near the limit-value $l_i + m_j$. Then we have to consider:*

$$\mathop{\mathbf{L}}_{n=\infty}\left[\frac{F_{11}(n)(l_1+m_1)+F_{12}(n)(l_1+m_2)+F_{21}(n)(l_2+m_1)+F_{22}(n)(l_2+m_2)}{F_{11}(n)+F_{12}(n)+F_{21}(n)+F_{22}(n)}\right].$$

It is clear, however, that

$$\left.\begin{aligned} F_{11}(i) + F_{12}(i) &= f_1(i) + c_1 \\ F_{21}(i) + F_{22}(i) &= f_2(i) + c_2 \end{aligned}\right\}\ \left\{\begin{aligned} F_{11}(i) + F_{21}(i) &= g_1(i) + d_1 \\ F_{12}(i) + F_{22}(i) &= g_2(i) + d_2 \end{aligned}\right.$$

where c_1, c_2, d_1, d_2, are constants independent of i. We accordingly obtain:

$$\mathop{\mathbf{L}}_{n=\infty}\left[\frac{F_{11}(n)(l_1 + m_1) + F_{12}(n)(l_1 + m_2) + F_{21}(n)(l_2 + m_1) + F_{22}(n)(l_2 + m_2)}{F_{11}(n) + F_{12}(n) + F_{21}(n) + F_{22}(n)}\right]$$

$$= \mathop{\mathbf{L}}_{n=\infty}\left\{\frac{[F_{11}(n)+F_{12}(n)]l_1+[F_{21}(n)+F_{22}(n)]l_2+[F_{11}(n)+F_{21}(n)]m_1+[F_{12}(n)+F_{22}(n)]m_2}{F_{11}(n) + F_{12}(n) + F_{21}(n) + F_{22}(n)}\right\}$$

* We have defined averageability for sequences with only two limit values. The extension to sequences with any finite number of limit-values is obvious (see page 19).

$$= \underset{n=\infty}{\mathbf{L}} \frac{[f_1(n)+c_1]l_1+[f_2(n)+c_2]l_2}{f_1(n)+c_1+f_2(n)+c_2} + \underset{n=\infty}{\mathbf{L}} \frac{[g_1(n)+d_1]m_1+[g_2(n)+d_2]m_2}{g_1(n)+d_1+g_2(n)+d_2}$$

$$= \underset{n=\infty}{\mathbf{L}} \left[\frac{f_1(n)l_1+f_2(n)l_2}{f_1(n)+f_2(n)}\right] + \underset{n=\infty}{\mathbf{L}} \left[\frac{g_1(n)m_1+g_2(n)m_2}{g_1(n)+g_2(n)}\right] = s+t.$$

Thus it is seen that the requirements* of page 2 are satisfied by our definition. The extension of the definition to the case of sequences with any finite number of limit values is obvious.

Definition: A sequence having k limit values, $l_1, l_2, \cdots l_k$, shall be called averageable, and have s for its value, if

$$\underset{i=\infty}{\mathbf{L}} \left[\frac{\sum_{n=1}^{n=k} f_n(i) l_n}{\sum_{n=1}^{n=k} f_n(i)}\right] = s.$$

It can be easily verified that Theorem I applies to this extended definition.

But we can generalize the notion of averageability even to cases where the sequence has an infinite number of limit-values. Let us consider a *reducible* sequence, and let us write:

$$\begin{array}{lllllll}
(E) \equiv (E^{(0)}) \equiv s_0, & s_1, & s_2, & \cdots s_n, & \cdots \\
(E^{(1)}) \equiv l_0^{(1)}, & l_1^{(1)}, & l_2^{(1)}, & \cdots l_n^{(1)}, & \cdots \\
(E^{(2)}) \equiv l_0^{(2)}, & l_1^{(2)}, & l_2^{(2)}, & \cdots l_n^{(2)}, & \cdots \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\
(E^{(k)}) \equiv l_0^{(k)}, & l_1^{(k)}, & l_2^{(k)}, & \cdots l_n^{(k)}, & \cdots,
\end{array}$$

where the sequence $(E^{(j)})$ consists of the limit values of the sequence $(E^{(j-1)})$. Since the sequence is assumed to be reducible, there exists a k such that $(E^{(k+1)}) \equiv 0$. Then (E) is reducible of order k, and $(E^{(k)})$ has only a finite number of elements.

* Requirement (v) is satisfied by each definition considered.

Let us assume that our sequence is reducible of order k, and that $(E^{(k)})$ has for its elements $l_0^{(k)}, l_1^{(k)}, \cdots l_p^{(k)}$. If now we choose e sufficiently small, all but a finite number of the $l_i^{(k-1)}$ will fall in the intervals $| l_i^{(k-1)} - l_p^{(k)} | < e$, $p = 0, 1, 2, \cdots p$. Suppose that the finite number of $l_i^{(k-1)}$ which do not fall in any of these intervals is p_1, and call them, $m_1^{(k-1)}$, $m_2^{(k-1)}$, $\cdots m_{p_1}^{(k-1)}$. We can choose $e_1 < e$, so small that only a finite number, p_2, of the $l_i^{(k-2)}$ do not fall in any of the intervals above, or in the intervals $| l_i^{(k-2)} - m_p^{(k-1)} | < e_1$, $p = 1, 2, \cdots p_1$. Call this finite set of limit points $m_1^{(k-2)}, \cdots m_{p_2}^{(k-2)}$. We can repeat this process until we reach the sequence (E), which will have only a finite number of elements outside of all the intervals considered.

*Definition: A reducible sequence shall be called averageable, with s for its sum, provided**

$$s = \underset{e=0}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}} \left\{ \frac{\sum_{j=1}^{j=k} \sum_{i=1}^{i=p_{k-j+1}} f_i^{(j)}(n, e) m_i^{(j)}}{\sum_{j=1}^{j=k} \sum_{i=1}^{i=p_{k-j+1}} f_i^{(j)}(n, e)} \right\} = \underset{e=0}{\mathrm{L}} F(e)$$

exists.

In this general definition it is convenient to distinguish between different kinds of limit points. Let us suppose that $f_i(n, e)$ corresponds to the limit point m_i, and let us assume that the following limit

$$a_i = \underset{n=\infty}{\mathrm{L}} \frac{f_i(n, e)}{\sum_{j=1}^{j=k} \sum_{i=1}^{i=p_{k-j+1}} f_i(n, e)}$$

exists for every i. We shall call m_i a *weak* or a *strong* limit point according as a_i is or is not equal to zero. We may then state the following proposition:

THEOREM 2: *A reducible averageable sequence with a finite number of strong limit points is averageable independent of e.*

* We have put $m_i^{(k)} = l_i^{(k)}$ for the sake of uniformity.

For simplicity consider the case where the reducibility is of order 2. The strong limit points are then either of the first or of the second order. There is only a finite number of strong limit points of order 2, and a finite number of strong limit points of order 1. Let m be the total number of strong limit points. Since for the remaining limit points $a_i = 0$, we have

$$F(e) = \underset{n=\infty}{\mathrm{L}}\left[\frac{f_1(n, e)l_1 + f_2(n, e)l_2 + \cdots + f_m(n, e)l_m}{\sum\limits_{i=1}^{m} f_i(n, e)}\right].$$

If we now choose $e' < e$, the values of the coefficients of the strong limit points are unaffected. Hence $F(e') = F(e)$, and our theorem is proved.

THEOREM 3: *A reducible averageable sequence with a finite number of strong limit points is Cesàro-summable of order* 1; *and the two values obtained are equal.*

We lay off e_i intervals about the limit points of order $k-i+1$, $(i = 1, 2, \cdots k)$ as on page 20, and we thus have for $n > N$, if e is the largest of the e_i,

$$\left.\begin{array}{l} |\, l_1 - s_i' \,| < e,\ i = 1, 2, \cdots f_1(n, e) \\ |\, l_2 - s_i'' \,| < e,\ i = 1, 2, \cdots f_2(n, e) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ |\, l_p - s_i^{(p)} \,| < e,\ i = 1, 2, \cdots f_p(n, e) \end{array}\right\} \begin{array}{l}\text{where } s_i^{(j)} \text{ are those } s_i \\ \text{which fall in the } e\text{-in-} \\ \text{terval about } l_j. \end{array}$$

We have accordingly:

$$|\,(f_1 l_1 + f_2 l_2 + \cdots + f_p l_p) - [(s_1' + \cdots + s_{f_1}') + \cdots \\ + (s_1^{(p)} + \cdots + s_{f_p}^{(p)})]\,| < (f_1 + f_2 \cdots + f_p)e.$$

Since

$$(s_1' + s_2' + \cdots + s_{f_1'}) + \cdots + (s_1^{(p)} + \cdots + s_{f_p}^{(p)}) \\ = s_{m+1} + s_{m+2} + \cdots + s_{m+q}$$

where $q = f_1 + f_2 + \cdots + f_p$, and m is sufficiently large, we have:

$$\left| \frac{f_1 l_1 + f_2 l_2 + \cdots + f_p l_p}{q} - \frac{s_{m+1} + s_{m+2} + \cdots + s_{m+q}}{q} \right| < e.$$

Hence

$$\underset{n=\infty}{\mathrm{L}} \left[\frac{f_1 l_1 + f_2 l_2 + \cdots + f_p l_p}{f_1 + f_2 + \cdots + f_p} \right] = \underset{q=\infty}{\mathrm{L}} \left[\frac{s_m + s_{m+1} + \cdots + s_{m+q}}{q} \right]$$

provided either limit exists. By Theorem 2, the left-hand limit exists independently of e; accordingly the right-hand limit exists; that is, the given sequence is summable (C_1).

In practice, the following proposition, a corollary of the theorem just proved, will be found useful:

Corollary: *If for some positive integer* k, *and for every positive integer* $i \leq k$, *the sequence* $s_i, s_{i+k}, s_{i+2k}, \cdots$ *converges, then the sequence* $s_1, s_2, \cdots$ *is summable* (C_1).

Let us take as an example the sequence

$$\begin{aligned} s_i &= i \log \left(1 + \frac{1}{i} \right), & i \text{ odd} \\ &= 0, & i \text{ even} \end{aligned}$$

to which it is not easy to apply the formula

$$\underset{n=\infty}{\mathrm{L}} \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

We see, however, that the two sequences

$$\left. \begin{array}{l} s_1, s_3, \cdots \\ s_2, s_4, \cdots \end{array} \right\}$$

converge; hence the given sequence is summable (C_1).

§ 4. PRODUCT DEFINITIONS

In dealing directly with sequences, the Cauchy-product* of two series does not appear to be entirely natural. Even in the case of convergent sequences, a more natural definition of product is close to hand. In fact, if s and t are the respective sums of two convergent sequences,

$$\begin{cases} s_0, \ s_1, \ s_2, \ \cdots \ s_n, \ \cdots \\ t_0, \ t_1, \ t_2, \ \cdots \ t_n, \ \cdots, \end{cases}$$

then it follows from a fundamental theorem of limits that

$$\underset{n=\infty}{\mathbf{L}} s_n t_n = st.$$

We are accordingly led† to propose the following

Definition: *The natural-product of two sequences,*

$$s_0, \ s_1, \ s_2, \ \cdots \ s_n, \ \cdots; \quad t_0, \ t_1, \ \cdots \ t_n, \ \cdots,$$

is the sequence: $s_0 t_0, \ s_1 t_1, \ \cdots \ s_n t_n, \ \cdots$.

We may then state the obvious proposition:

THEOREM: *The natural-product of two convergent sequences, whose values are s and t respectively, is itself convergent*; *and its value is st.*

If we compare this theorem with the corresponding theorem‡ for the Cauchy-product, it will be seen at once that the natural-product is of superior value to the Cauchy-product, in the case of convergent sequences of constant terms. In the case of sequences which are not convergent, however, the natural-product can play no part. For consider the simple example,

* See page 5.

† Baire: *Cours D'analyse*, t. I.

‡ Theorem B, page 6.

$$\begin{cases} s = 1, 0, 1, 0, \cdots \\ t = 1, 0, 1, 0, \cdots \\ w = 1, 0, 1, 0, \cdots, \end{cases}$$

where the sequence whose value is w is the *natural-product* of the two sequences whose values are s and t respectively. Here $s = t = w = \frac{1}{2}$, and accordingly $w \neq st$. We are consequently led to generalize the definition for the product of two sequences. Let us consider again the two sequences

$$\begin{cases} s_0, s_1, s_2, \cdots s_n, \cdots \\ t_0, t_1, t_2, \cdots t_n, \cdots \end{cases}$$

and let us form the array:

$$\begin{array}{cccccc} s_0t_0, & s_0t_1, & s_0t_2, & \cdots & s_0t_n, & \cdots \\ s_1t_0, & s_1t_1, & s_1t_2, & \cdots & s_1t_n, & \cdots \\ s_2t_0, & s_2t_1, & s_2t_2, & \cdots & s_2t_n, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ s_nt_0, & s_nt_1, & s_nt_2, & \cdots & s_nt_n, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Definition: The sequence formed by following the successive lines which form squares with the boundaries of the array, i. e.,

$$s_0t_0;\ s_0t_1,\ s_1t_1,\ s_1t_0;\ s_0,t_2,\ s_1t_2,\ s_2t_2,\ s_2t_1,\ s_2t_0;\ \cdots,$$

shall be called the square-product of the two sequences.

We shall now prove the following theorem:

THEOREM 4: *The square-product of two averageable sequences is averageable, and its value is equal to the product of their values.*

Let the given sequences be

$$\begin{cases} s = s_0,\ s_1,\ \cdots\ s_n,\ \cdots \\ t = t_0,\ t_1,\ \cdots\ t_n,\ \cdots; \end{cases}$$

we wish to prove that the sequence

$$s_0t_0;\ s_0t_1,\ s_1t_1,\ s_1t_0;\ s_0t_2,\ s_1t_2,\ s_2t_2,\ s_2t_1,\ s_2t_0;\ \cdots$$

is averageable, and that its value is st. We shall assume* that the sequence (s) has the two limit-values l_1, l_2, and that the sequence (t) has the two limit-values m_1, m_2. The only limit-values of the product sequence are then: l_1m_1, l_1m_2, l_2m_1 and l_2m_2. We are given

$$s = \underset{n=\infty}{\mathbf{L}}\left[\frac{f_1(n)l_1 + f_2(n)l_2}{f_1(n) + f_2(n)}\right],$$

$$t = \underset{n=\infty}{\mathbf{L}}\left[\frac{g_1(n)l_1 + g_2(n)l_2}{g_1(n) + g_2(n)}\right],$$

and we wish to consider:

$$\underset{n=\infty}{\mathbf{L}}\left[\frac{F_{11}(n)l_1m_1 + F_{12}(n)l_1m_2 + F_{21}(n)l_2m_1 + F_{22}(n)l_2m_2}{F_{11}(n) + F_{12}(n) + F_{21}(n) + F_{22}(n)}\right],$$

where $F_{ij}(n)$ is the number of elements of the product sequence near l_im_j. If we pick n elements from the product sequence, we observe:

$$\left.\begin{aligned} F_{11}(n) &= f_1(n)g_1(n) + k_{11} \\ F_{12}(n) &= f_1(n)g_2(n) + k_{12} \end{aligned}\right\}\ \left\{\begin{aligned} F_{21}(n) &= f_2(n)g_1(n) + k_{21} \\ F_{22}(n) &= f_2(n)g_2(n) + k_{22}, \end{aligned}\right.$$

where k_{ij} are constants independent of n. We have, accordingly,

$$\underset{n=\infty}{\mathbf{L}}\left[\frac{F_{11}(n)l_1m_1 + F_{12}(n)l_1m_2 + F_{21}(n)l_2m_1 + F_{22}(n)l_2m_2}{F_{11}(n) + F_{12}(n) + F_{21}(n) + F_{22}(n)}\right]$$

$$= \underset{n=\infty}{\mathbf{L}}\left[\frac{\begin{aligned}&[f_1(n)g_1(n) + k_{11}]l_1m_1 + [f_1(n)g_2(n) + k_{12}]l_1m_2 \\ &\qquad + [f_2(n)g_1(n) + k_{21}]l_2m_1 + [f_2(n)g_2(n) + k_{22}]l_2m_2\end{aligned}}{\begin{aligned}&f_1(n)g_1(n) + k_{11} + f_1(n)g_2(n) + k_{12} + f_2(n)g_1(n) \\ &\qquad + k_{21} + f_2(n)g_2(n) + k_{22}\end{aligned}}\right]$$

* The proof for the general case is precisely similar.

$$= \underset{n=\infty}{\mathrm{L}} \left[\frac{f_1(n)g_1(n)l_1m_1 + f_1(n)g_2(n)l_1m_2 + f_2(n)g_1(n)l_2m_1 + f_2(n)g_2(n)l_2m_2}{f_1(n)g_1(n) + f_1(n)g_2(n) + f_2(n)g_1(n) + f_2(n)g_2(n)} \right]$$

$$= \underset{n=\infty}{\mathrm{L}} \left[\frac{[f_1(n)l_1 + f_2(n)l_2][g_1(n)m_1 + g_2(n)m_2]}{[f_1(n) + f_2(n)][g_1(n) + g_2(n)]} \right] = st.$$

For example, the square-product of the sequences

$$\begin{cases} s = 1, 0, 1, 0, \cdots \\ t = 1, 0, 1, 0, \cdots, \end{cases}$$

is

$w = 1;\ 0, 0, 0;\ 1, 0, 1, 0, 1;\ 0, 0, 0, 0, 0, 0, 0;\ 1, 0, 1, 0, 1, 0, 1, 0;\ \cdots.$
If we choose m terms of this sequence, and let $(2n)^2$ be the largest square of an even integer less than or equal to m, so that

$$m = (2n)^2 + k, \quad 0 \leqq k < 8n + 4,$$

we get:

$$w = \underset{n=\infty}{\mathrm{L}} \left\{ \frac{[1+3+\cdots+(2n-1)]1 + [m-(1+\cdots+2n-1)]\cdot 0}{m} \right\}$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{n^2}{m} = \underset{n=\infty}{\mathrm{L}} \frac{n^2}{4n^2 + k}$$

$$= \tfrac{1}{4}.$$

Thus it is verified that $w = s \cdot t$.

Although it is true that the natural-product is better adapted to convergent sequences than the Cauchy-product, and that the square-product is better suited for averageable sequences, it must be remembered that in analysis the things that arise frequently are not sequences of constant terms, but rather *series* of *variable terms*, notably power series. In the case of power series, the Cauchy-product is certainly more valuable; for if we multiply two such series according to the Cauchy scheme, we obtain the same result which is given by multiplying the two series as if they were polynomials, thus:

$$\begin{cases} u(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots + u_n x^n + \cdots \\ v(x) = v_0 + v_1 x + v_2 x^2 + v_3 x^3 + \cdots + v_n x^n + \cdots \end{cases}$$

$$w(x) = u(x) \cdot v(x) = u_0 v_0 + (u_0 v_1 + u_1 v_0) x + (u_0 v_2 + u_1 v_1 + u_2 v_0) x^2 + \cdots.$$

Furthermore, to this symbolic advantage is added the theoretical one which is contained in the following theorem, due to Cesàro,* which is a generalization of Theorem B.

THEOREM (J): *The Cauchy-product of two Cesàro-summable series, of orders p and q, and of values s and t respectively, is itself Cesàro-summable of order at most $p + q + 1$, and its value is st.*

In certain special cases, we can slightly improve upon the results of Cesàro's theorem. Thus, if two series are convergent (i. e., summable of order 0), their product must be summable of order at most 1. If, however, one of these series converges absolutely, then the product-series is convergent,† as has already been stated.‡ Similarly, the Cauchy-product of two Cesàro-summable series, one of order r, the other convergent, is summable (C_{r+1}); if the convergent series happens to be absolutely convergent, however, the product can be shown to be summable (C_r).

THEOREM 5: *The Cauchy-product of a Cesàro-summable series of order r by an absolutely convergent series, is itself Cesàro-summable of order r.*

Let

$$\begin{cases} s_n = u_0 + u_1 + \cdots + u_n, \\ t_n = v_0 + v_1 + \cdots + v_n, \\ w_n = u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0, \\ y_n = w_0 + w_1 + \cdots + w_n. \end{cases}$$

* Cesàro: *Bull. des Sciences math.*, t. XIV, 1890.

† Mertens, *Journal de Crelle*, t. 79, p. 182.

‡ P. 5, supra.

$$\left\{\begin{aligned} Y_n &= y_0 \frac{r(r+1)\cdots(r+n-1)}{n!} + y_1 \frac{r(r+1)\cdots(r+n-2)}{(n-1)!} \\ &\qquad + \cdots + y_{n-2}\frac{r(r+1)}{2!} + y_{n-1}\cdot r + y_n, \\ T_n &= t_0 \frac{r(r+1)\cdots(r+n-1)}{n!} + t_1 \frac{r(r+1)\cdots(r+n-2)}{(n-1)!} \\ &\qquad + \cdots + t_{n-2}\frac{r(r+1)}{2!} + t_{n-1}\cdot r + t_n, \end{aligned}\right.$$

$$(r, n) = \frac{r(r+1)\cdots(r+n-1)}{n!}; \quad t_n = \frac{T_n}{(r, n)}.$$

We assume:

$$\underset{n=\infty}{\mathbf{L}} s_n = s, \quad \underset{n=\infty}{\mathbf{L}} [|u_0| + |u_1| + \cdots + |u_n|] = A,$$

$$\underset{n=\infty}{\mathbf{L}} \frac{T_n}{\dfrac{(r+1)\cdots(r+n)}{n!}} = t,$$

and we wish to prove:

$$\underset{n=\infty}{\mathbf{L}} \frac{Y_n}{\dfrac{(r+1)\cdots(r+n)}{n!}} = s\cdot t.$$

Proof:

Lemma: If

$$\underset{n=\infty}{\mathbf{L}} \frac{T_n}{\dfrac{(r+1)\cdots(r+n)}{n!}} = t, \quad \text{then} \quad \underset{n=\infty}{\mathbf{L}} \frac{T_n - T_{n-p}}{\dfrac{(r+1)\cdots(r+n)}{n!}} = 0,$$

$$p = 1, 2, \cdots p.$$

For

$$\underset{n=\infty}{\mathbf{L}} \left[\frac{T_n}{\dfrac{(r+1)\cdots(r+n)}{n!}} - \frac{T_{n-p}}{\dfrac{(r+1)\cdots(r+n)}{n!}} \right]$$

$$= \mathop{\mathbf{L}}_{n=8}\left[\frac{T_n}{\dfrac{(r+1)\cdots(r+n)}{n!}} - \frac{T_{n-p}}{\dfrac{(r+1)\cdots(r+n-p)}{(n-p)!}} \cdot \frac{\dfrac{(r+1)\cdots(r+n-p)}{(n-p)!}}{\dfrac{(r+1)\cdots(r+n)}{n!}}\right]$$

$$= \mathop{\mathbf{L}}_{n=\infty}\left[\frac{T_n}{\dfrac{(r+1)\cdots(r+n)}{n!}} - \frac{T_{n-p}}{\dfrac{(r+1)\cdots(r+n-p)}{(n-p)!}} \cdot \frac{n(n-1)\cdots(n-p+1)}{(r+n)\cdot(r+n-1)\cdots(r+n-p+1)}\right] = t - t\cdot 1 = 0,$$

Now

$$y_n = u_0v_0 + (u_0v_1 + u_1v_0) + (u_0v_2 + u_1v_1 + u_2v_0) + \cdots + (u_0v_n + u_1v_{n-1} + \cdots + u_{n-1}v_1 + u_nv_0)$$

$$= u_0(v_0 + v_1 + \cdots + v_n) + u_1(v_0 + v_1 + \cdots + v_{n-1}) + \cdots + u_{n-1}(v_0 + v_1) + u_nv_0,$$

$$y_n = u_0t_n + u_1t_{n-1} + \cdots + u_{n-1}t_1 + u_nt_0.$$

$$Y_n = u_0t_0\frac{r(r+1)\cdots(r+n-1)}{n!} + (u_0t_1 + u_1t_0)\frac{r(r+1)\cdots(r+n-2)}{(n-1)!} + \cdots + (u_0t_{n-2} + \cdots + u_{n-2}t_0)\frac{r(r+1)}{2!} + (u_0t_{n-1} + \cdots + u_{n-1}t_0)r + (u_0t_n + \cdots + u_nt_0),$$

$$Y_n = u_0T_n + u_1T_{n-1} + \cdots + u_{n-1}T_1 + u_nT_0,$$

$$\therefore\ Y_{2n} = u_0T_{2n} + u_1T_{2n-1} + \cdots + u_{2n-1}T_1 + u_{2n}T_0,$$

Let

$$R \equiv \left| \frac{\dfrac{Y_{2n}}{(r+1)\cdots(r+2n)}}{(2n)!} - (u_0+u_1+\cdots+u_n)\frac{T_n}{\dfrac{(r+1)\cdots(r+n)}{n!}} \right|$$

$$\equiv \left| \frac{(u_0T_{2n}+u_1T_{2n-1}+\cdots+u_{2n-1}T_1+u_{2n}T_0)}{\dfrac{(r+1)(r+2)\cdots(r+2n)}{(2n)!}} - (u_0+u_1+\cdots+u_n)\frac{T_n}{\dfrac{(r+1)\cdots(r+n)}{n!}} \right|$$

$$R \overline{<} \left[|u_0| \left| \frac{T_{2n}}{(r+1,2n)} - \frac{T_n}{(r+1,n)} \right| + |u_1| \left| \frac{T_{2n-1}}{(r+1,2n)} - \frac{T_n}{(r+1,n)} \right| + \cdots + |u_q| \left| \frac{T_{2n-q}}{(r+1,2n)} - \frac{T_n}{(r+1,n)} \right| \right]$$

$$+ \left[|u_{q+1}| \left| \frac{T_{2n-q-1}}{(r+1,2n)} - \frac{T_n}{(r+1,n)} \right| + \cdots + |u_n| \left| \frac{T_n}{(r+1,2n)} - \frac{T_n}{(r+1,n)} \right| \right]$$

$$+ \left[|u_{n+1}| \left| \frac{T_{n-1}}{r+1,2n} \right| + \cdots + |u_{2n}| \left| \frac{T_0}{(r+1,2n)} \right| \right]$$

$$\overline{<} M_1 + M_3 + |u_{q+1}| \left\{ \left| \frac{T_{2n-q-1}}{r+1,2n-q-1} \right| + \left| \frac{T_n}{(r+1,n)} \right| \right\} + \cdots + |u_n| \left\{ \left| \frac{T_n}{(r+1,n)} \right| + \left| \frac{T_n}{(r+1,n)} \right| \right\},$$

where M_1, M_2 and M_3 stand respectively for the expressions in the first, second and third brackets above.

$$R < M_1 + M_3 + \{|u_{q+1}| + \cdots + |u_n|\} B,$$

since

$$\left| \frac{T_m}{(r+1, m)} \right| < \frac{B}{2} \text{ for all } m.$$

Also

$$M_3 < \{|u_{n+1}| + \cdots + |u_{2n}|\} B.$$

Now as to M_1,

$$\left| \frac{T_{2n-p}}{(r+1, 2n)} - \frac{T_n}{(r+1, n)} \right| < \left| \frac{T_{2n}}{(r+1, 2n)} - \frac{T_n}{(r+1, n)} \right|$$

$$+ \left| \frac{T_{2n-p}}{(r+1, 2n)} - \frac{T_{2n}}{(r+1, 2n)} \right|; \quad p = 0, 1, 2, \cdots q$$

$$< \frac{\delta}{2(A+B)} + \frac{\delta}{2(A+B)}, \text{ if } n > N;$$

$$\therefore R < \{|u_0| + |u_1| + \cdots + |u_q|\} \frac{e}{A+B}$$

$$+ \{|u_{q+1}| + \cdots + |u_n| + \cdots + |u_{2n}|\} B, \text{ if } n > N.$$

Now choose q so large, that

$$|u_{q+1}| + \cdots + |u_{2n}| < \frac{e}{A+B}, \quad q > Q \text{ for } \textit{all } n.$$

Moreover, $|u_0| + \cdots + |u_q| < A$ for *all* q.

$$\therefore R < \frac{eA + eB}{A+B} = e.$$

Thus

$$\underset{n=\infty}{\mathbf{L}} \frac{Y_{2n}}{(r+1, 2n)} = s \cdot t.$$

Similarly

$$\underset{n=\infty}{\mathbf{L}} \frac{Y_{2n+1}}{(r+1, 2n+1)} = s \cdot t.$$

The theorem is now proved.

In the case of power series, then, both the symbolic advantage and the theoretical importance of Theorems J and 5 lead naturally to the Cauchy-product. This advantage does not appear, however, in case of sequences which do not correspond to power series,—for example, in Fourier's series; in this case, the square-product may be of greater service than the Cauchy-product. We should observe, however, that while the square-product may justly replace the Cauchy definition of multiplication, in certain cases; the definition of averageability has the disadvantage of presupposing the knowledge of the limit-values; and these are not always easy to determine even in the case of sequences of constant terms.

§ 5. ON CERTAIN POSSIBLE DEFINITIONS OF SUMMABILITY

Cauchy has proved* the following theorem, which we shall show is equivalent to Theorem C.

THEOREM K: *If* $u_n > 0$ *and*

$$\underset{n=\infty}{\mathrm{L}} \frac{u_{n+1}}{u_n} = l, \quad \textit{then} \quad \underset{n=\infty}{\mathrm{L}} u_n^{1/n} = l.$$

Let

$$\frac{u_{n+1}}{u_n} = t_{n+1}, \quad u_0 = 1,$$

then

$$u_n = t_1 t_2 \cdots t_n.$$

Accordingly, whenever

$$\underset{n=\infty}{\mathrm{L}} t_n = t,$$

then

$$\underset{n=\infty}{\mathrm{L}} (t_1 t_2 \cdots t_n)^{1/n} = t,$$

provided $t_n > 0$; and the last equation may be written

$$\log t = \underset{n=\infty}{\mathrm{L}} \left(\frac{\log t_1 + \log t_2 + \cdots + \log t_n}{n} \right).$$

And if we finally write $\log t_n = s_n$, we obtain the result that

$$\underset{n=\infty}{\mathrm{L}} \frac{s_1 + s_2 + \cdots + s_n}{n} = s$$

whenever

$$\underset{n=\infty}{\mathrm{L}} s_n = s.$$

This statement is, however, precisely Theorem C. We see accordingly that Theorems C and K are equivalent, by means of the substitution

* *Cours d'Analyse*: Oeuvres de Cauchy (2° serie), Vol. 3, pt. 3.

$$\frac{u_{n+1}}{u_n} = t_{n+1} = e^{s_{n+1}}.$$

Let us make the further substitution $s_n = r_n \varphi_n$, and observe that the variables s_n and r_n on each side of this equation approach the same limit, provided

$$\mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1.$$

We may accordingly replace Theorem C, which we have just obtained again, by the following theorem:

THEOREM 6: *If*

$$\mathop{\mathrm{L}}_{n=\infty} r_n = r, \quad \textit{and} \quad \mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1,$$

then

$$\mathop{\mathrm{L}}_{n=\infty}\left[\frac{\varphi_1 r_1 + \varphi_2 r_2 + \cdots + \varphi_n r_n}{n}\right] = r.$$

If we put a further restriction on the sequence φ_n we can broaden the requirement on the sequence r_n. In fact, we may say:

THEOREM 7: *If*

$$\mathop{\mathrm{L}}_{n=\infty} \frac{r_1 + r_2 + \cdots + r_n}{n} = r,$$

and

$$\mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1$$

monotonically, then*

* That the theorem is not true in general, when

$$\mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1$$

not monotonically, follows from the example:

$$r_n = (-1)^{n+1} \log n, \quad \varphi_n = 1 + (-1)^{n+1} \frac{1}{\log n}, \quad n \neq 1, \quad \varphi_1 = 1.$$

Here

$$\mathop{\mathrm{L}}_{n=\infty} \frac{r_1 + \cdots + r_n}{n} = 0, \quad \mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1$$

not monotonically;

$$\mathop{\mathrm{L}}_{n=\infty} \frac{\varphi_1 r_1 + \cdots + \varphi_n r_n}{n} = 1.$$

$$\underset{n=\infty}{\mathrm{L}} \frac{\varphi_1 r_1 + \varphi_2 r_2 + \cdots + \varphi_n r_n}{n} = r.$$

The proof of this theorem follows at once from the following theorem due to Hardy;* for a proof of which see page 85.

THEOREM L: *If Σc_n is a divergent series of positive terms, then*

$$\underset{n=\infty}{\mathrm{L}} \frac{c_0 s_0 + c_1 s_1 + \cdots + c_n s_n}{n + 1} = \underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_n}{n + 1},$$

provided that the second limit exists and either

(*a*) *c_n steadily decreases,*

(*b*) *c_n steadily increases, subject to the condition*

$$nc_n < (c_0 + c_1 + \cdots + c_n)K,$$

where K is a fixed number.

We shall now show that Theorem 7 is a special case of Theorem L. In the first place, since

$$\underset{n=\infty}{\mathrm{L}} \varphi_n = 1,$$

it follows from Theorem C that

$$\underset{n=\infty}{\mathrm{L}} \frac{\varphi_1 + \varphi_2 + \cdots + \varphi_n}{n} = 1,$$

and accordingly,

$$\underset{n=\infty}{\mathrm{L}} \frac{\varphi_1 r_1 + \varphi_2 r_2 + \cdots + \varphi_n r_n}{n}$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{\varphi_1 r_1 + \varphi_2 r_2 + \cdots + \varphi_n r_n}{\varphi_1 + \varphi_2 + \cdots + \varphi_n} \cdot \frac{\varphi_1 + \varphi_2 + \cdots + \varphi_n}{n}$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{\varphi_1 r_1 + \varphi_2 r_2 + \cdots + \varphi_n r_n}{\varphi_1 + \varphi_2 + \cdots + \varphi_n}.$$

We may now apply Theorem L directly, by identifying φ_n

* *Quarterly Journal*, Vol. 38 (1907), p. 269. Hardy proves a more general theorem of which this is a special case; the first part of the general theorem has been first proved, however, by Cesàro, as Hardy himself states. See Cesàro: *Bull. des Sciences math.* (2), t. 13, 1889, p. 51.

with c_n. If φ_n decreases monotonically, the condition of the first part of Theorem L is fulfilled; if φ_n increases monotonically, we have:

$$\varphi_1 + \varphi_2 + \cdots + \varphi_n > n\varphi_1,$$

or

$$\frac{\varphi_1 + \varphi_2 + \cdots + \varphi_n}{n} \geq \varphi_1 = \frac{1}{k} > \frac{\varphi_n}{K}, \quad K > k,$$

so that

$$\varphi_n < K\frac{(\varphi_1 + \varphi_2 + \cdots + \varphi_n)}{n},$$

which is precisely the second requirement of Theorem L. Hence the truth of Theorem 7 is established.

We can deduce an interesting consequence from Theorem 7, and say, in the language of § 4,

Theorem 8: *The natural product of two sequences, one of which is summable of order* 1, *the other monotonically convergent, is summable of order* 1; *and the value of the product sequence is equal to the product of the values of the two given sequences.*

Let s_n and t_n be the two given sequences,

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_1 + s_2 + \cdots + s_n}{n} = s, \quad \mathop{\mathrm{L}}_{n=\infty} t_n = t,$$

monotonically. We first suppose that $t \neq 0$, and form the sequence t_n/t, so that

$$\mathop{\mathrm{L}}_{n=\infty} \frac{t_n}{t} = 1$$

monotonically. Accordingly, by Theorem 7,

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_1\dfrac{t_1}{t} + s_2\dfrac{t_2}{t} + \cdots + s_n\dfrac{t_n}{t}}{n} = s$$

or

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_1t_1 + s_2t_2 + \cdots + s_nt_n}{n} = st.$$

If $t = 0$, we form the sequence $1 + t_n$, so that

$$\mathop{\mathrm{L}}_{n=\infty}(1 + t_n) = 1$$

monotonically; consequently, by Theorem 7,

$$s = \mathop{\mathrm{L}}_{n=\infty} \frac{s_1(1 + t_1) + s_2(1 + t_2) + \cdots + s_n(1 + t_n)}{n}$$

$$= \mathop{\mathrm{L}}_{n=\infty} \frac{s_1 + s_2 + \cdots + s_n}{n} + \mathop{\mathrm{L}}_{n=\infty} \frac{s_1 t_1 + s_2 t_2 + \cdots + s_n t_n}{n}$$

and accordingly,

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_1 t_1 + s_2 t_2 + \cdots + s_n t_n}{n} = 0.$$

Let us now return to Theorem 6, and base upon it the following definition:

Definition: *The sequence shall be said to be φ-summable, and to have the value s, provided*

$$\begin{cases} \mathop{\mathrm{L}}\limits_{n=\infty} \dfrac{s_1\varphi_1 + s_2\varphi_2 + \cdots + s_n\varphi_n}{n} = s. \\ \mathop{\mathrm{L}}\limits_{n=\infty} \varphi_n = 1. \end{cases}$$

It is natural to ask for the relation between φ-summability and Cesàro-summability. In general it will be possible to find a sequence φ_n which will give a more general definition than that of Cesàro-summability of order 1. We can however restrict the sequence φ_n so as to make the two definitions equivalent; and we may state the following theorem:

THEOREM 9: *If*

$$\mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1$$

monotonically, then whenever either of the two definitions—φ-summability or Cesàro-summability of order 1*—gives a value to a given sequence, so will the other, and the two values will be the same.*

If we choose any specific sequence $\bar{\varphi}_n$, subject to the condition

$$\mathop{\mathrm{L}}_{n=\infty} \bar{\varphi}_n = 1$$

monotonically, then it follows at once from Theorem 7 that if a sequence is summable of order 1, it is also φ-summable for the particular $\bar{\varphi}_n$. Let us now suppose, conversely, that the sequence $s_1, s_2, \cdots s_n, \cdots$ is φ-summable for $\bar{\varphi}_n$, i. e.,

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_1\bar{\varphi}_1 + s_2\bar{\varphi}_2 + \cdots + s_n\bar{\varphi}_n}{n} = s.$$

This amounts to saying that the sequence $(s_n\varphi_n)$ is Cesàro-summable of order 1. Let us now apply Theorem 7, making $r_n = s_n\varphi_n$, and $\varphi_n = 1/\bar{\varphi}_n$. Since

$$\mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1$$

monotonically, then

$$\mathop{\mathrm{L}}_{n=\infty} \bar{\varphi}_n = 1$$

monotonically, and

$$s = \mathop{\mathrm{L}}_{n=\infty} \left[\frac{s_1\bar{\varphi}_1\varphi_1 + s_2\bar{\varphi}_2\varphi_2 + \cdots + s_n\bar{\varphi}_n\varphi_n}{n}\right] = \mathop{\mathrm{L}}_{n=\infty} \frac{s_1 + s_2 + \cdots + s_n}{n},$$

i. e., the given sequence is Cesàro-summable of order 1.

If we assume that

$$\mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1$$

non-monotonically, then Theorem 7 may no longer apply, as is shown by the following example:

$$s_i = (-1)^{i+1} \log i \begin{cases} \varphi_1 = 1 \\ \varphi_i = 1 + (-1)^{i+1} \dfrac{1}{\log i}, \quad i = 2, 3, \cdots, \end{cases}$$

so that

$$\begin{aligned} s_1\varphi_1 &= 0 \\ s_i\varphi_i &= 1 + (-1)^{i+1} \log i = 1 + s_i, \quad i = 2, 3, \cdots. \end{aligned}$$

Now

$$\underset{n=\infty}{\mathrm{L}} \frac{s_1 + s_2 + \cdots + s_n}{n} = \underset{n=\infty}{\mathrm{L}} \frac{\log 1 - \log 2 + \cdots \pm \log n}{n} = 0^*$$

and

$$\underset{n=\infty}{\mathrm{L}} \varphi_n = 1$$

non-monotonically.

If Theorem 7 were true, φ_n non-monotonic, we should have

$$\underset{n=\infty}{\mathrm{L}} \frac{s_1\varphi_1 + s_2\varphi_2 + \cdots + s_n\varphi_n}{n} = 0;$$

whereas,

$$\underset{n=\infty}{\mathrm{L}} \frac{s_1\varphi_1 + \cdots + s_n\varphi_n}{n} = \underset{n=\infty}{\mathrm{L}} \frac{\log 1 + (1 + s_2) + \cdots + (1 + s_n)}{n}$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{n-1}{n} + \underset{n=\infty}{\mathrm{L}} \frac{s_2 + \cdots + s_n}{n} = 1.$$

Returning now to the monotonic φ-definition, we observe that if we take $\varphi_n \equiv 1$, we obtain Cesàro-summability of order 1. Taking

$$\varphi_n = \log\left(1 + \frac{1}{n}\right)^n,$$

we obtain:

$$(6) \quad s = \underset{n=\infty}{\mathrm{L}} \left[\frac{s_1 \log 2 + s_2 \log(1 + \frac{1}{2})^2 + s_3 \log(1 + \frac{1}{3})^3 + \cdots + s_n \log\left(1 + \frac{1}{n}\right)^n}{n}\right].$$

* $\underset{n=\infty}{\mathrm{L}} \frac{1}{2n} \sum_{i=1}^{2n} (-1)^{i+1} \log i = \frac{1}{2} \underset{n=\infty}{\mathrm{L}} \log\left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}\right)^{1/n}$

$$= \frac{1}{2} \underset{n=\infty}{\mathrm{L}} \log u_n^{1/n} = 0$$

since

$$\underset{n=\infty}{\mathrm{L}} \frac{u_{n+1}}{u_n} = 1.$$

Also

$$\underset{n=\infty}{\mathrm{L}} \frac{1}{2n+1} \sum_{i=1}^{2n+1} (-1)^{i+1} \log i = 0 + \underset{n=\infty}{\mathrm{L}} \frac{\log(2n+1)}{2n+1} = 0.$$

Since

$$\underset{n=\infty}{\mathrm{L}} \log\left(1+\frac{1}{n}\right)^n = 1$$

monotonically, however, it follows that this definition is equivalent to Cesàro-summability of order 1, or (what amounts to the same thing) equivalent to Hölder-summability of order 1. If we now write

$$t_n = \frac{s_1 + s_2 + \cdots + s_n}{n},$$

so that

$$nt_n - (n-1)t_{n-1} = s_n,$$

we may repeat the process for the sequence t_n, obtaining

$$\underset{n=\infty}{\mathrm{L}} \left[\frac{t_1 \log 2 + t_2 \log(1+\frac{1}{2})^2 + \cdots + t_n \log\left(\frac{n+1}{n}\right)^n}{n}\right]$$

$$= \underset{n=\infty}{\mathrm{L}} \left[\frac{s_1 \log 2 + (s_1+s_2)\log\frac{3}{2} + \cdots + (s_1+s_2+\cdots+s_n)\log\frac{n+1}{n}}{n}\right]$$

$$(7)\quad s = \underset{n=\infty}{\mathrm{L}} \left[\frac{s_1 \log\frac{n+1}{1} + s_2 \log\frac{n+1}{2} + \cdots + s_n \log\frac{n+1}{n}}{n}\right].$$

Since (6) is equivalent to the Hölder-summability of s_n of order 1, it follows that (7) is equivalent to Hölder-summability of t_n of order 1, i. e., with Hölder-summability of s_n of order 2.

Let us now return to our definition of φ-summability, and repeat the process for another function $\psi(n)$, where

$$\underset{n=\infty}{\mathrm{L}} \psi(n) = 1.$$

Writing

$$t_n = \frac{\varphi(1)s_1 + \varphi(2)s_2 + \cdots + \varphi(n)s_n}{n},$$

we obtain

$$\underset{n=\infty}{\mathrm{L}} \left[\frac{\psi(1)t_1 + \psi(2)t_2 + \psi(n)t_n}{n} \right]$$

$$(8) \quad = \underset{n=\infty}{\mathrm{L}} \left\{ \frac{s_1\,\varphi(1)\left\{\psi(1) + \frac{\psi(2)}{2} + \cdots + \frac{\psi(n)}{n}\right\} + s_2\varphi(2)\left\{\frac{\psi(2)}{2} + \cdots + \frac{\psi(n)}{n}\right\} + \cdots + s_n\varphi(n)\frac{\psi(n)}{n}}{n} \right\}.$$

Now, if

$$\underset{n=\infty}{\mathrm{L}}\, \varphi(n) = \underset{n=\infty}{\mathrm{L}}\, \psi(n) = 1,$$

then

$$\underset{n=\infty}{\mathrm{L}}\, r_n = \underset{n=\infty}{\mathrm{L}} \frac{\varphi_1 + \varphi_2 + \cdots + \varphi_n}{n} = 1,$$

and

$$\underset{n=\infty}{\mathrm{L}} \left\{ \frac{\psi(1)r_1 + \psi(2)r_2 + \cdots + \psi(n)r_n}{n} \right\}$$

$$(9) \quad = \underset{n=\infty}{\mathrm{L}} \left\{ \frac{\varphi(1)\left\{\psi(1) + \frac{\psi(2)}{2} + \cdots + \frac{\psi(n)}{n}\right\} + \varphi(2)\left\{\frac{\psi(2)}{2} + \cdots + \frac{\psi(n)}{n}\right\} + \cdots + \varphi(n)\frac{\psi(n)}{n}}{n} \right\} = 1.$$

Instead of taking

$$\underset{n=\infty}{\mathrm{L}}\, \varphi(n) = \underset{n=\infty}{\mathrm{L}}\, \psi(n) = 1,$$

we shall assume more generally that (9) is satisfied, and take as our definition,

$$
(10)\quad \begin{cases} \displaystyle \mathop{\mathrm{L}}_{n=\infty}\left\{\frac{\begin{aligned} &s_1\varphi(1)\left[\psi(1)+\frac{\psi(2)}{2}+\cdots+\frac{\psi(n)}{n}\right] \\ &\qquad +s_2\varphi(2)\left[\frac{\psi(2)}{2}+\cdots+\frac{\psi(n)}{n}\right]+\cdots \\ &\qquad\qquad +s_n\varphi(n)\left[\frac{\psi(n)}{n}\right]\end{aligned}}{n}\right\} = s, \\[2ex] \displaystyle \mathop{\mathrm{L}}_{n=\infty}\left\{\frac{\begin{aligned} &\varphi(1)\left[\psi(1)+\frac{\psi(2)}{2}+\cdots+\frac{\psi(n)}{n}\right] \\ &+\varphi(2)\left[\frac{\psi(2)}{2}+\cdots+\frac{\psi(n)}{n}\right]+\cdots+\varphi(n)\left[\frac{\psi(n)}{n}\right]\end{aligned}}{n}\right\} = 1. \end{cases}
$$

If $\varphi(n) \equiv \psi(n) \equiv 1$, we obtain:

$$
\begin{aligned}
s &= \mathop{\mathrm{L}}_{n=\infty}\left\{\frac{s_1\left[1+\frac{1}{2}+\cdots+\frac{1}{n}\right]+s_2\left[\frac{1}{2}+\cdots+\frac{1}{n}\right]+\cdots+s_n\left[\frac{1}{n}\right]}{n}\right\} \\
&= \mathop{\mathrm{L}}_{n=\infty}\left\{\frac{s_1+\frac{s_1+s_2}{2}+\frac{s_1+s_2+s_3}{3}+\cdots+\frac{s_1+s_2+s_n}{n}}{n}\right\} \\
&= \mathop{\mathrm{L}}_{n=\infty}\left[\frac{t_1+t_2+\cdots+t_n}{n}\right] \quad \text{where} \quad t_n = \frac{s_1+s_2+\cdots+s_n}{n},
\end{aligned}
$$

which is Hölder summability of order 2.

If

$$
\varphi(n) \equiv 2n, \quad \psi(n) = \frac{1}{n+1},
$$

we obtain:

$$
s = \mathop{\mathrm{L}}_{n=\infty}\left[\frac{\begin{aligned} &s_1 2\left[\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{n(n+1)}\right] \\ &+s_2\cdot 2.2\left[\frac{1}{2.3}+\cdots+\frac{1}{n(n+1)}\right]+\cdots+s_n\cdot 2n\left[\frac{1}{n(n+1)}\right]\end{aligned}}{n}\right]
$$

$$= \underset{n=\infty}{\mathbf{L}} \left[\frac{s_1 \dfrac{2n}{n+1} + s_2 \dfrac{(2n-1)}{n+1} + \cdots + s_n \cdot \dfrac{2}{(n+1)}}{n} \right]$$

$$= \underset{n=\infty}{\mathbf{L}} \left[\frac{ns_1 + (n-1)s_2 + \cdots + s_n}{\dfrac{n(n+1)}{2\,!}} \right],$$

which is Cesàro-summable of order 2.

If we put

$$\varphi_n \equiv 1, \quad \psi_n = n \log\left(1 + \frac{1}{n}\right),$$

we obtain:

$$s = \underset{n=\infty}{\mathbf{L}} \left[\frac{\begin{array}{r} s_1\left\{\log 2 + \log\left(1 + \frac{1}{2}\right) + \cdots + \log\left(1 + \frac{1}{n}\right)\right\} \\ +s_2\left\{\log\left(1+\frac{1}{2}\right) + \cdots + \log\left(1+\frac{1}{n}\right)\right\} + \cdots \\ +s_n\left\{\log\left(1+\frac{1}{n}\right)\right\} \end{array}}{n} \right]$$

$$= \underset{n=\infty}{\mathbf{L}} \left[\frac{\begin{array}{l} s_1\{(\log 2 - \log 1) + (\log 3 - \log 2) + \cdots \\ \qquad + (\log(n+1) - \log n)\} + \cdots + s_n\left\{\log \frac{n+1}{n}\right\} \end{array}}{n} \right]$$

$$= \underset{n=\infty}{\mathbf{L}} \left[\frac{s_1 \log \dfrac{n+1}{1} + s_2 \log \dfrac{n+1}{2} + \cdots + s_n \log \dfrac{n+1}{n}}{n} \right]$$

which is (7).

We have thus seen that the definitions of φ-summability and (10) include some of the specific definitions which we have already discussed. One might naturally ask, however, whether these general definitions themselves may be of any use. One use immediately presents itself, as can be seen in the following example.

It is desired to know whether the series given by

$$\begin{cases} s_i = \dfrac{1}{i \log\left(1 + \dfrac{1}{i}\right)}, & i = \text{odd} \\ \quad = 0 \qquad\qquad\qquad , & i = \text{even} \end{cases}$$

is summable* according to Cesàro's definition; and if so, its value is required. To determine this directly from Cesàro's definition requires some manipulation. If we choose, however,

$$\varphi_i = i \log\left(1 + \frac{1}{i}\right),$$

we obtain

$$\underset{n=\infty}{\mathrm{L}} \frac{s_1\varphi_1 + s_2\varphi_2 + \cdots + s_n\varphi_n}{n} = \underset{n=\infty}{\mathrm{L}} \frac{1 + 0 + 1 + 0 + \cdots + 0 \text{ or } 1}{n}$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{\dfrac{n}{2} \text{ or } \dfrac{n}{2} + 1}{n} = \tfrac{1}{2}.$$

And since

$$\underset{n=\infty}{\mathrm{L}} \varphi_n = 1$$

monotonically, it follows that

$$\underset{n=\infty}{\mathrm{L}} \frac{s_1 + s_2 + \cdots + s_n}{n} = \tfrac{1}{2}.$$

This example leads us to formulate the following proposition, which is of practical importance:

THEOREM 10: *To test a given sequence for Cesàro-summability of order* 1, *any convenient* φ_n *may be chosen, provided*

$$\underset{n=\infty}{\mathrm{L}} \varphi_n = 1$$

monotonically.

Similarly we may sometimes simplify our calculations in testing for Cesàro-summability of order 2, if we can find a suitable φ_n and ψ_n.

* This example has been already considered from another standpoint. See p. 22.

We might now proceed to generalize to p-functions, and show that the resulting generalizations would include all of Cesàro's and Hölder's definitions. And from what has preceded, it is easily seen that if we take all the p-functions equal to unity, we shall obtain all of Hölder's forms; while by a suitable choice of these p-functions, all of the Cesàro-forms might also be obtained. But though the process is quite clearly defined, the algebraic details become too complicated to carry this work any further. The fact, however, that we may use, as a definition of summability, the limit of an expression in which the coefficients of the s_i are not specifically named, but are given in terms of functions satisfying certain conditions, suggests a more general view of summability, which we shall proceed to develop in the next article.

§ 6. DEFINITIONS OF EVALUABILITY

We have now considered a large number of definitions of summability. It is natural to ask whether all those definitions do not have some common properties. Excepting for the moment Borel's definitions, to which we shall return later, we can say that all* the definitions of summability which we have considered have the following properties in common:

If $a_i(n)$ represents the coefficient of s_i in any of the expressions whose limit gives rise to one of the definitions of summability, then:

(i) $$\underset{n=\infty}{\mathrm{L}}\, a_i(n) = 0, \quad \text{for fixed } i,$$

(ii) $$\underset{n=\infty}{\mathrm{L}}\, [a_1(n) + a_2(n) + \cdots + a_n(n)] = 1,$$

(iii) $$a_i(n) \geq^{\dagger} 0 \text{ for all } i \text{ and } n.$$

That properties (i) and (iii) are common to all* of the definitions under consideration is easily verified. We proceed to show that the same is true of property (ii). Beginning with Cesàro-summability of order r, we shall show that the sum of the coefficients of the numerator, divided by the denominator, is identically equal to unity. For this purpose we write:

$$(1 - x)^{-(r+1)} = (1 + x + x^2 + x^3 \cdots + x^n + \cdots)(1 - x)^{-r}.$$

Equating the coefficients of x^n on each side of this identity, we obtain:

$$\frac{(r+1)(r+2)\cdots(r+n)}{n!} \equiv 1 + r + \frac{r(r+1)}{2!} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!},$$

* We exclude also definition (10).

† The equality sign occurs in the case of convergence.

so that:

$$\frac{\dfrac{r(r+1)(r+2)\cdots(r+n-1)}{n!}+\cdots+\dfrac{r(r+1)}{2!}+r+1}{\dfrac{(r+1)(r+2)\cdots(r+n)}{n!}} \equiv 1.$$

Turning now to Hölder's definitions, we observe that for order 1, the sum of the coefficients of the s_i is identically equal to unity—this being in fact a special case of the case just considered. Suppose now that $h_1, h_2, \cdots h_n$ are the coefficients of Hölder's definition of order p, so that

$$\underset{n=\infty}{\mathbf{L}}\,[h_1 s_1 + h_2 s_2 + \cdots + h_n s_n] = s.$$

If we assume that $h_1 + h_2 + \cdots + h_n \equiv 1$ for order p, we obtain for order $p + 1$, putting

$$s_n = \frac{t_1 + t_2 + \cdots + t_n}{n},$$

$$\underset{n=\infty}{\mathbf{L}}\left[h_1 t_1 + h_2\frac{t_1+t_2}{2} + \cdots + h_n\frac{t_1+\cdots+t_n}{n}\right]$$
$$= \underset{n=\infty}{\mathbf{L}}\left[t_1\left(h_1 + \frac{h_2}{2} + \cdots + \frac{h_n}{n}\right) + \cdots + t_n\frac{h_n}{n}\right]$$

and the sum of the coefficients becomes

$$\left[h_1 + \frac{h_2}{2} + \cdots + \frac{h_n}{n}\right] + \left[\frac{h_2}{2} + \cdots + \frac{h_n}{n}\right] + \cdots + \frac{h_n}{n}$$
$$\equiv h_1 + h_2 + \cdots + h_n \equiv 1.$$

Thus the proof of (ii) for Hölder's definitions is completed by mathematical induction.

Let us now consider formula (7). We shall show that

$$\underset{n=\infty}{\mathbf{L}}\, u_n = 1,$$

where

$$u_n = \frac{\log \frac{n}{1} + \log \frac{n}{2} + \cdots + \log \frac{n}{n-1}}{n} = \log \left(\frac{n}{1} \cdot \frac{n}{2} \cdots \frac{n}{n-1} \right)^{\frac{1}{n}}.$$

If

$$v_n = \frac{n}{1} \cdot \frac{n}{2} \cdots \frac{n}{n-1} = \frac{1}{(n-1)!} n^{n-1},$$

then

$$\frac{v_{n+1}}{v_n} = \left(1 + \frac{1}{n}\right)^n.$$

Hence

$$\underset{n=\infty}{\mathrm{L}}\, v_n^{1/n} = \underset{n=\infty}{\mathrm{L}} \frac{v_{n+1}}{v_n} = e.$$

Accordingly,

$$\underset{n=\infty}{\mathrm{L}}\, u_n = \underset{n=\infty}{\mathrm{L}} \log v_n^{1/n} = 1.$$

Finally since we have assumed in the φ-definition that

$$\underset{n=\infty}{\mathrm{L}}\, \varphi(n) = 1,$$

it follows that

$$\underset{n=\infty}{\mathrm{L}} \frac{\varphi(1) + \varphi(2) + \cdots + \varphi(n)}{n} = 1$$

by Theorem c.

Thus it is seen that all* of these definitions have properties (i) to (iii) in common. We can accordingly generalize our notion of summability by stating a definition in terms of these properties themselves.

Definition: *A series shall be said to be A-evaluable,*† *and to have the sum s whenever the following conditions are fulfilled:*

---

* Except definition (10)

† We shall hereafter use the term *evaluable* in the case of definitions in terms of *properties* of general functional coefficients of the s_i; the word *summable* we shall retain for concrete definitions with specific coefficients.

$$
(A)\quad\begin{cases}
\text{(i)} & \underset{n=\infty}{\mathrm{L}}\, a_i(n) = 0, \text{ for fixed } i,\\
\text{(ii)} & \underset{n=\infty}{\mathrm{L}}\, [a_1(n) + a_2(n) + \cdots + a_n(n)] = 1,\\
\text{(iii)} & a_i(n) \geq 0,\\
\text{(iv)} & \underset{n=\infty}{\mathrm{L}}\, [a_1(n)s_1 + a_2(n)s_2 + \cdots + a_n(n)s_n] = s.
\end{cases}
$$

We shall now justify this definition by proving the following theorem:

THEOREM II: *If a series is convergent then it is A-evaluable.**

By (iv) we may write:

$$
\text{(v)}\qquad\begin{cases}
[a_1(n) + a_2(n) + \cdots + a_n(n)] + r_n \equiv 1,\\
\underset{n=\infty}{\mathrm{L}}\, r_n = 0.
\end{cases}
$$

Now, by (v),

$$
\begin{aligned}
&| a_1(n)s_1 + a_2(n)s_2 + \cdots + a_n(n)s_n - s |\\
&\qquad \equiv | \{a_1(n)s_1 + a_2(n)s_2 + \cdots + a_n(n)s_n\}\\
&\qquad - (a_1(n) + a_2(n) + \cdots + a_n(n) + r_n)s |\\
&\qquad \overline{\leq} \, | a_1(n)(s_1 - s) + a_2(n)(s_2 - s) + \cdots + a_p(n)(s_p - s) |\\
&\qquad + | a_{p+1}(n)(s_{p+1} - s) + \cdots + a_n(n)(s_n - s) | + | r_n s | .
\end{aligned}
$$

Since the series is convergent, we can choose i so large that

$$
|s_i - s| < \eta, \qquad i > p.
$$

Let l be the largest of the numbers $| s_i - s |$, for $i = 1, 2, \cdots p$. We have, then,

$$
\begin{aligned}
&| a_1(n)s_1 + a_2(n)s_2 + \cdots + a_n(n)s_n - s |\\
&\qquad \overline{\leq} \, \{a_1(n) | s_1 - s | + \cdots + a_p(n) | s_p - s|\}
\end{aligned}
$$

* Theorem II obtains if condition (iii) is replaced by the broader condition:

$$
| a_1(n) | + | a_2(n) | + \cdots + | a_n(n) | < K.
$$

$$+ \{a_{p+1}(n)|s_{p+1} - s| + \cdots + a_n(n)|s_n - s|\} + |r_n s|$$
$$< \{a_1(n)+\cdots+a_p(n)\}l+\{a_{p+1}(n)+\cdots+a_n(n)\}\eta+|r_n s|$$
$$< \delta l + \eta + |r_n s|, \quad n > N^*$$
$$< \frac{e}{3}+\frac{e}{3}+\frac{e}{3} \text{ by (v)}, \quad \text{if } \delta = \frac{e}{3l}, \quad \eta = \frac{e}{3}.$$
$$= e.$$

Hence

$$\mathop{\mathrm{L}}_{n=\infty} [a_1(n)s_1 + a_2(n)s_2 + \cdots + a_n(n)s_n] = s.$$

Our definition of A-evaluability is now justified.

The question naturally suggests itself as to whether for a sequence (s_n) which diverges to $+\infty$,

$$\mathop{\mathrm{L}}_{n=\infty} \sum_{i=1}^{n} a_i(n)s_i = +\infty.$$

The answer, which is in the affirmative, is embodied in the following theorem:

THEOREM IIA: *If*

$$\mathop{\mathrm{L}}_{n=\infty} s_n = +\infty,$$

and conditions (i), (ii), (iii) *are satisfied, then*

$$\mathop{\mathrm{L}}_{n=\infty} \sum_{i=1}^{n} a_i(n)s_i = +\infty.$$

By hypothesis, $s_n > N$, $n > m$. Hence

$$\sigma_n = \sum_{i=1}^{n} a_i(n)s_i = \sum_{i=1}^{m} a_i(n)s_i + \sum_{m+1}^{n} a_i(n)s_i$$
$$> \sum_{i=1}^{m} a_i(n)s_i + N \sum_{i=m+1}^{n} a_i(n).$$

* By (i), $[a_1(n) + \cdots + a_p(n)] < \delta$, $n > N$, p having been chosen first, and then held fast. By (iii), $[a_{p+1}(n) + \cdots + a_n(n)] < [a_1(n) + \cdots + a_n(n)] < 1$ by (ii).

Since

$$\underset{n=\infty}{\mathrm{L}}\left[\sum_{i=1}^{m} a_i(n)s_i + N\sum_{i=m+1}^{n} a_i(n)\right] = N,$$

it follows that

$$\text{Minimum} \underset{n=\infty}{\mathrm{L}} \sigma_n \geq N;$$

and since N is an arbitrary number,

$$\underset{n=\infty}{\mathrm{L}} \sigma_n = +\infty.$$

We have seen that the generalized definition includes a large number of the specific definitions of summability which we have considered. But we see now that if we take any functions whatever for $a_i(n)$, subject merely to the restrictions (i), (ii) and (iii), we may obtain a possible definition of summability. Thus, we may take as our definition, for example,

$$(11)\quad s = \underset{n=\infty}{\mathrm{L}}\left[\frac{s_1 + \frac{s_2}{2} + \cdots + \frac{s_n}{n}}{\log n}\right] = \underset{n=\infty}{\mathrm{L}}\left[\frac{s_1 + \frac{1}{2}s_2 + \cdots + \frac{1}{n}s_n}{1 + \frac{1}{2} + \cdots + \frac{1}{n}}\right].$$

This formula is of interest to us, since it affords an example of a definition which is broader than Cesàro-summability of order 1, and yet perhaps not so general as that of order 2. For since $1/n$ steadily decreases, it follows from Theorem 8 that formula (11) gives a value to all series that are Cesàro-summable of order 1, and that these values are the same for both definitions. That (11) is really more general than summability of order 1 follows from the example $1 - 3 + 5 - 7 + \cdots$. This series is not summable of order 1, since

$$\underset{n=\infty}{\mathrm{L}} \frac{u_n}{n} \neq 0;$$

however we obtain from (11), for the corresponding sequence, $s_n = (-1)^{n+1}n$,

$$\underset{n=\infty}{\mathrm{L}}\left[\frac{1-1+1-1\cdots\pm 1}{\log n}\right]=\underset{n=\infty}{\mathrm{L}}\left[\frac{1 \text{ or } 0}{\log n}\right]=0.$$

Nevertheless, (11) is probably not equivalent to summability of order 2, as the following reasoning suggests. A necessary condition that a series give a result by (11) is

$$\underset{n=\infty}{\mathrm{L}}\frac{u_n}{n\log n}=0.*$$

This is not, however, a necessary condition for summability of order 2†—so that we might find a series for which

$$\underset{n=\infty}{\mathrm{L}}\frac{u_n}{n\log n}\neq 0,$$

which is nevertheless summable of order 2.

We have seen that the A-definition includes most of the cases of summability which we have discussed, but we have been obliged to omit Borel's definitions. In order to include the Borel-mean-definition, we shall now generalize Theorem 11, as well as the definition which we have based upon it. Replacing $a_i(n)$ by $a_i(\alpha)$, where α may be independent of n, Theorem (11) may be stated in a more general form:

THEOREM 12: *From the conditions*:

$$*0=\underset{n=\infty}{\mathrm{L}}\left[\frac{s_1+\frac{1}{2}s_2+\cdots+\frac{1}{n+1}s_{n+1}}{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}-\frac{s_1+\frac{1}{2}s_2+\cdots+\frac{1}{n}s_n}{1+\frac{1}{2}+\cdots+\frac{1}{n}}\right]=\underset{n=\infty}{\mathrm{L}}\frac{s_n}{n\log n}.$$

$$\underset{n=\infty}{\mathrm{L}}\frac{u_n}{n\log n}=\underset{n=\infty}{\mathrm{L}}\frac{s_n-s_{n-1}}{n\log n}=0.$$

† A necessary condition for summability of order 2 is

$$\underset{n=\infty}{\mathrm{L}}\frac{u_n}{n^2}=0.$$

See p. 10.

$$\begin{cases} \text{(i)} & \underset{\alpha \doteq \infty}{\mathrm{L}}\, a_i(\alpha) = 0 \text{ for fixed } i, \\ \text{(ii)} & \underset{n=\infty}{\mathrm{L}}\, [a_1(\alpha) + a_2(\alpha) + \cdots + a_n(\alpha)] \equiv 1, \\ \text{(iii)} & a_i(\alpha) \geq 0, \\ \text{(iv)} & \underset{n=\infty}{\mathrm{L}}\, s_n = s, \end{cases}$$

may be deduced the result:

$$\underset{\alpha \doteq \infty}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}}\, [a_1(\alpha)s_1 + a_2(\alpha)s_2 + \cdots + a_n(\alpha)s_n] = s.$$

We shall first show that

$$\underset{n=\infty}{\mathrm{L}}\, [a_1(\alpha)s_1 + a_2(\alpha)s_2 + \cdots + a_n(\alpha)s_n]$$

exists for every definite α. Taking a definite value of α,

$$\begin{aligned} |a_n(\alpha)s_n + a_{n+1}(\alpha)s_{n+1} + \cdots + a_{n+p}(\alpha)s_{n+p}| \\ &\lesseqgtr a_n(\alpha)|s_n| + \cdots + a_{n+p}(\alpha)|s_{n+p}| \\ &< \frac{e}{A} \cdot A \text{ by (ii)} \qquad \big((n > N, \text{ any } p)\big) \\ &= e. \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} a_n(\alpha)s_n$$

converges for every value of α. Since

$$\sum_{n=1}^{\infty} a_n(\alpha)s_n$$

has a sense, we may write:

$$\left|\sum_{n=1}^{\infty} a_n(\alpha)s_n - s\right| \equiv \left|\sum_{n=1}^{\infty} a_n(\alpha)s_n - \sum_{n=1}^{\infty} a_n(\alpha) \cdot s\right| \text{ by (ii)}$$

$$= \left| \sum_{n=1}^{\infty} a_n(\alpha)(s_n - s) \right| \overline{<} \left| \sum_{n=1}^{m-1} a_n(\alpha)(s_n - s) \right| + \left| \sum_{n=m}^{\infty} a_n(\alpha)(s_n - s) \right|$$

$$< H \sum_{n=1}^{m-1} a_n(\alpha) + e,$$

since $|s_n - s| < e$, $n \geq m$, and $|s_n - s| < H$, $n < m$ by (iv). Since, however,

$$\underset{\alpha \doteq \infty}{\mathrm{L}} \sum_{n=1}^{m-1} a_n(\alpha) = 0$$

by (i), it follows that:

$$\text{Maximum} \underset{\alpha=\infty}{\mathrm{L}} \left| \sum_{n=1}^{\infty} a_n(\alpha) s_n - s \right| \overline{<} e + \text{Maximum} \underset{\alpha \doteq \infty}{\mathrm{L}} H \sum_{n=1}^{m-1} a_n(\alpha) = e.$$

Since e is arbitrarily small, the maximum limit on the left must be zero, and therefore the actual limit is zero, i. e.,

$$\underset{\alpha=\infty}{\mathrm{L}} \sum_{n=1}^{\infty} a_n(\alpha) s_n = s.$$

It is readily seen that Borel's mean-definition satisfies conditions (i) to (iii) of Theorem 12. For we have, in satisfaction of condition (i),

$$\underset{\alpha=\infty}{\mathrm{L}} \frac{\alpha^n}{e^\alpha} = 0;$$

that condition (ii) is satisfied follows since

$$\underset{n=\infty}{\mathrm{L}} \left[\frac{1 + \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \cdots + \frac{\alpha^n}{n!}}{e^\alpha} \right] \equiv 1;$$

and finally, since $\alpha^n/e^\alpha > 0$ for $\alpha > 0$, it follows that (iii) is fulfilled.

We might accordingly generalize our definition of evaluability, to include Borel's mean-definition, by using the hypotheses (i) to (iii) of Theorem 12 as a basis. It turns out, however, that we may generalize Theorem 12 still further, and that we can accordingly obtain a still more general definition of evaluability.

Let us take as coefficients of the s_i functions of *both* n and α, and write:

$$\begin{cases} \text{(i)} & \underset{n=\infty}{\mathrm{L}}\, a_i(\alpha,\ n) \equiv 0, \\ \text{(ii)} & \underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(\alpha,\ n) \equiv 1, \\ \text{(iii)} & a_i(\alpha,\ n) \geq 0. \end{cases}$$

If now these conditions are fulfilled for a fixed value of α, and if

$$\underset{n=\infty}{\mathrm{L}}\, s_n = s,$$

it follows from Theorem II, that

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(\alpha,\ n) s_i = s.$$

Since this limit exists for every value of α, under our hypothesis, we may write:

$$\text{(iv)} \qquad \underset{\alpha=\alpha_0}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(\alpha,\ n) s_i = s,$$

and a definition that readily suggests itself, even when the series is not convergent, is that conditions (i) to (iv) be fulfilled.

We have demanded at the very start, however, that every definition should satisfy certain fundamental requirements, which we have enumerated on page 2, and while the definition proposed does fulfil the first two of those requirements, as we have just seen, it does not fulfil the third requirement* without further restrictions on the coefficients.†

Our third fundamental demand was that when the series $u_0 + u_1 + u_2 + \cdots + u_n + \cdots$ has the value s, then the series $u_1 + u_2 + \cdots + u_n + \cdots$ must have the value $s - u_0$;

* The same is true, of course, for the A-definition; we have deferred the similar considerations for that case, since they may be included under this more general one.

† It is obvious that the fourth and fifth requirements are also fulfilled.

or stated in terms of sequences, if $s_n = u_0 + u_1 + \cdots + u_n$, when the sequence $s_0, s_1, s_2, \cdots s_n, \cdots$ has the value s, then the sequence $s_1 - u_0, s_2 - u_0, \cdots s_n - u_0, \cdots$ has the value $s - u_0$. If we *assume*, for the moment, that whenever either one of the two sequences

$$s_0,\ s_1,\ s_2,\ \cdots s_n,\ \cdots$$
$$s_1,\ s_2,\ \cdots s_n,\ \cdots$$

has the value s, the other does also; then we shall satisfy our third requirement if we prove that whenever $s_1, s_2, s_3, \cdots s_n, \cdots$ has the value s, then $s_1 - u_0, s_2 - u_0, s_3 - u_0, \cdots s_n - u_0, \cdots$ has the value $s - u_0$. Now this it is easy to prove. For we have by iv, p. 55,

$$\underset{\alpha=\infty}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(\alpha, n)(s_i - u_0) \equiv \underset{\alpha=\infty}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(\alpha, n)s_i - u_0 = s - u_0$$

by (ii), p. 55.

It remains then to consider under what restrictions we can justify our assumption that the two sequences

$$s_0,\ s_1,\ s_2,\ \cdots s_n,\ \cdots$$
$$s_1,\ s_2,\ \cdots s_n,\ \cdots$$

always have a value together. To get an idea as to the nature of the condition which we shall have to add, let us consider, for concreteness, what happens in the case of Borel's mean-definition.

Using the notation of page 12, we have:

$$\begin{cases} s(\alpha) = s_0 + s_1\dfrac{\alpha}{1} + s_2\dfrac{\alpha^2}{2!} + \cdots + s_n\dfrac{\alpha^n}{n!} + \cdots, \\[2ex] s'(\alpha) = s_1 + s_2\dfrac{\alpha}{1} + \cdots + s_n\dfrac{\alpha^{n-1}}{(n-1)!} + \cdots, \end{cases}$$

$$s'(\alpha) - s(\alpha) = u_1 + u_2\frac{\alpha}{1} + u_3\frac{\alpha^2}{2!} + \cdots + u_n\frac{\alpha^{n-1}}{(n-1)!} + \cdots,$$

Borel's definition being

$$s = \underset{\alpha=\infty}{\mathrm{L}}\left[\frac{s(\alpha)}{e^\alpha}\right].$$

If we assume* that $\underset{a=\infty}{\mathrm{L}}\, s(\alpha) = \infty$,

we have an indeterminate form, so that

$$\underset{a=\infty}{\mathrm{L}}\left[\frac{s(\alpha)}{e^{a}}\right] = \underset{a=\infty}{\mathrm{L}}\frac{s'(\alpha)}{e^{a}},$$

or

$$\underset{a=\infty}{\mathrm{L}}\frac{s'(\alpha) - s(\alpha)}{e^{a}} = 0,$$

which may be written,

$$\underset{a=\infty}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}}\, e^{-a}\left[u_1 + u_2\frac{\alpha}{1} + u_3\frac{\alpha^2}{2!} + \cdots + u_{n+1}\frac{\alpha^n}{n!}\right] = 0.$$

It is accordingly suggested that we assume, in general,

(v) $\underset{a=a_0}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}}\,[a_0(\alpha, n)u_1 + a_1(\alpha, n)u_2 + \cdots + a_n(\alpha, n)u_{n+1}] = 0.$

As a matter of fact, this condition is sufficient,† for, from (iv)

(iv) $\underset{a=a_0}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}}\,[a_0(\alpha, n)s_0 + a_1(\alpha, n)s_1 + \cdots + a_n(\alpha, n)s_n] = s$ and

adding (iv) and (v) we obtain

$$\underset{a=a_0}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}}\,[a_0(\alpha, n)s_1 + a_1(\alpha, n)s_2 + \cdots + a_n(\alpha, n)s_{n+1}] = s,$$

which proves that when the sequence $s_0, s_1, \cdots s_n, \cdots$ is evaluable to s, so is the sequence $s_1, s_2, \cdots s_n, \cdots$. By subtracting (v) from the last limit we show in the same way that when the sequence $s_1, s_2, \cdots s_n, \cdots$ is evaluable to s, so is the sequence $s_0, s_1, s_2, \cdots s_n, \cdots$. Thus, condition (v) causes our definition to satisfy the third requirement of page 2. If we wish to be able to drop any finite number of terms, we shall have to require a condition more general than (v), as we shall do in the following definition:

* This assumption is not essential, since our object is simply to arrive at a certain condition on the $a_i(\alpha, n)$.

† Condition (v) is not satisfactory since it is a condition on the sequence, as well as on $a_i(n, \alpha)$. It would be desirable to have on $a_i(n, \alpha)$ further restrictions, sufficient to cause (v) to hold for all sequences.

Definition: A series shall be said to be B-evaluable and to have the sum s whenever the following conditions are fulfilled:

$$B\begin{cases} \text{(i) } \underset{n=\infty}{\mathrm{L}}\, a_i(\alpha, n) = 0, \\ \text{(ii) } \underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(\alpha, n) \equiv 1, \\ \text{(iii) } a_i(\alpha, n) \geq 0, \\ \text{(iv) } \underset{\alpha=\alpha_0}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(\alpha, n) s_i = s, \\ \text{(v) } \underset{\alpha=\alpha_0}{\mathrm{L}}\ \underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(\alpha, n) u_{i+k} = 0, \quad k = 1, 2, \cdots p. \end{cases}$$

We have seen that this definition includes all of the definitions of summability which we have considered, except possibly the Borel-integral definition. We have not yet subjected this integral definition to the test of our fundamental requirements; let us now do this.

That requirements (i) and (ii) are satisfied follows from the following theorem:* If

$$\underset{n=\infty}{\mathrm{L}}\, s_n = s,$$

then

$$\int_0^\infty e^{-r} u(r)dr = s,$$

where

$$u(r) = u_0 + u_1\frac{r}{1} + u_2\frac{r^2}{2\,!} + \cdots + u_n\frac{r^n}{n\,!} + \cdots.$$

It is obvious, too, that requirements (iv) and (v) are satisfied. Let us accordingly limit our considerations to requirement (iii). With regard to this requirement we have the following state of affairs:†

* Hardy: *Quarterly Journal*, Vol. 35, p. 22; Bromwich, loc. cit., p. 269.

† The quotation is taken from Bromwich, loc. cit., p. 271. The first of the propositions was proved by Borel, loc. cit., p. 101; Hardy proved the second proposition by an example: *Quarterly Journal*, Vol. 35 (1903), p. 30.

"Any finite number of terms may be prefixed to a summable series, and the series will remain summable. . . . But the removal of even a single term from the beginning of the series may destroy the property of summability."

Inasmuch then as the integral-definition fails to satisfy one of our fundamental requirements, we are obliged to rule it out. In fact Borel himself ruled it out,* replacing it by *absolute summability.*† This definition does satisfy requirement (iii), as Borel proves,‡ and it obviously satisfies requirements (ii), (iv) and (v). Furthermore, Borel makes the statement‡ that convergent series are always absolutely summable. Hence it would follow that the definition of absolute summability is to be retained, since it seems to satisfy all of the fundamental requirements.

But Borel's statement that convergent series are always absolutely summable, is incorrect, as Hardy§ has shown by the following example:

$$\begin{cases} u_n = \dfrac{(-1)^i}{i}, & n = i^2, \\ u_n = 0, & n \textit{ not } \text{a square.} \end{cases}$$

In fact the series in question:

$$-1 + 0 + 0 + \tfrac{1}{2} + 0 + 0 + 0 + 0 - \tfrac{1}{3} + \cdots$$

is convergent, while

$$\int_0^\infty e^{-r}|u(r)|dr$$

is divergent. Thus, since absolute summability fails to satisfy

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* Loc. cit., p. 99.

† See p. 14.

‡ Loc. cit., p. 100.

§ Hardy, loc. cit.

the first fundamental requirement, this definition too cannot be retained.*

We have seen that the B-definition satisfies all of our fundamental requirements, and that it includes as special cases all of the proposed definitions of summability which satisfy those requirements. Our definition of B-summability is accordingly justified.

We proceed to the statement of the following definitions:

Definition 1: *A series shall be called abstractly-evaluable, and to have the value s, if the following conditions are fulfilled:*

$$(a)\qquad \underset{n=\infty}{\mathrm{L}}\,[a_1(n)s_1 + a_2(n)s_2 + \cdots + a_n(n)s_n] = s,$$

(*b*) *the fundamental requirements of page 2 are satisfied.*

Definition 2: *An abstractly-evaluable series of functions of a variable shall be called uniformly evaluable, if:*

$$(c)\qquad \begin{aligned} &\underset{n=\infty}{\mathrm{L}}\,[a_1(n)s_1(x) + a_2(n)s_2(x) + \cdots + a_n(n)s_n(x)] \\ &\qquad = \underset{n=\infty}{\mathrm{L}}\, f(x, n) = s(x) \end{aligned}$$

uniformly.

From these definitions follow at once several theorems.

THEOREM 13: *A uniformly evaluable series of continuous functions represents a continuous function.*†

For $f(x, n) = a_1(n)s_1(x) + \cdots + a_n(n)s_n(x)$ is a continuous function of x; and since

$$\underset{n=\infty}{\mathrm{L}}\, f(x, n) = s(x)$$

uniformly, it follows that $s(x)$ is continuous.

Similarly, we should obtain in the usual way, the following two propositions:

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* It is for this reason that we omit from further considerations the integral definition and the extended definitions given by Borel himself and by Le Roy. See p. 14, supra.

† The same proof applies when the continuity is with respect to some assemblage.

THEOREM 13A: *A sufficient condition that an abstractly-evaluable series of continuous functions represent a continuous function is that the related sequence, $f(x, n)$, have Dini's simple-uniform convergence.**

THEOREM 13B: *A necessary and sufficient condition that an abstractly-evaluable series of continuous functions define a continuous function is that $f(x, n)$ have Arzelà's quasi-uniform convergence.*†

THEOREM 14: *A uniformly evaluable series of continuous functions may be integrated term by term.*

We wish to prove in this case that

$$\int_a^b \underset{n=\infty}{\mathrm{L}} [a_1(n)s_1(x) + a_2(n)s_2(x) + \cdots + a_n(n)s_n(x)]dx$$
$$= \underset{n=\infty}{\mathrm{L}} \int_a^b [a_1(n)s_1(x) + a_2(n)s_2(x) + \cdots + a_n(n)s_n(x)]dx$$

or

$$\int_a^b \underset{n=\infty}{\mathrm{L}} f(x, n)dx = \underset{n=\infty}{\mathrm{L}} \int_a^b f(x, n)dx,$$

but this equation is precisely a statement of the theorem that a uniformly convergent sequence of continuous functions may be integrated term by term.

THEOREM 15: *If a series of continuous functions is convergent for all values of x in an interval, except possibly for $x = x_0$; and if two sets of functions $a_i(n)$, $b_i(n)$ render the series abstractly-evaluable at x_0, to the values s and t respectively; then, if the evaluability of each of the definitions is uniform in the interval, then $s = t$.*

Letting

$$f(x, n) = \sum_{i=0}^{n} a_i(n)s_i(x),$$

and

$$g(x, n) = \sum_{i=0}^{n} b_i(n)s_i(x),$$

---

* Dini: *Fundamenti per la teoretica delle Funzioni di variabili reali.* Pise, 1878, p. 103.

† Arzelà: *Memoires de Bologne,* 1899.

and remembering that since the series is convergent, $x \neq x_0$, it is true that

$$\mathop{\mathrm{L}}_{n=\infty} f(x, n) = \mathop{\mathrm{L}}_{n=\infty} g(x, n), \quad x \neq x_0,$$

we have from the uniformity,

$$\left.\begin{aligned} \mathop{\mathrm{L}}_{x=x_0} \mathop{\mathrm{L}}_{n=\infty} f(x, n) &= \mathop{\mathrm{L}}_{n=\infty} f(x_0, n) = s, \\ \mathop{\mathrm{L}}_{x=x_0} \mathop{\mathrm{L}}_{n=\infty} g(x, n) &= \mathop{\mathrm{L}}_{n=\infty} g(x_0, n) = ,t \end{aligned}\right\}$$

and hence $s = t$.

We may obviously state the preceding theorem in the following more general manner:

THEOREM 15A: *If a series of functions continuous on an assemblage* (E) *is convergent at all points of* (E), *except possibly at* $x = x_0$, *which is a limit point of* (E); *and if two sets of functions* $a_i(n)$, $b_i(n)$ *render the series abstractly-evaluable at* x_0, *to the values* s *and* t *respectively*; *then, if the evaluability of each of the definitions is uniform on* (E), *it follows that* $s = t$.

§ 7. APPLICATIONS

We shall first consider an application of the definition of abstract evaluability to integral equations, and we shall obtain a generalization* of a theorem due to Volterra.† Let us seek for a continuous solution of the integral equation,

$$u(x) = f(x) + \int_a^b K(x, \xi)u(\xi)d\xi,$$

where $K(x, y)$ is continuous,‡

$$\left\{\begin{matrix} a \overline{\gtrless} x \overline{\gtrless} b \\ a \overline{\gtrless} y \overline{\gtrless} b \end{matrix}\right\}$$

and $f(x)$ is continuous, $a \overline{\gtrless} x \overline{\gtrless} b$.

Following the method of Volterra, we shall form the iterated functions:

$$(12) \qquad \begin{cases} K_1(x, y) = K(x, y), \\ K_i(x, y) = \displaystyle\int_a^b K_1(x, \xi)K_{i-1}(\xi, y)d\xi. \end{cases}$$

Then

$$K_i(x, y) = \int_a^b K(x, \xi_1)K(\xi_1, \xi_2) \cdots K(\xi_{i-1}, y)d\xi_{i-1} \cdots d\xi_1$$

and

$$K_{i+j}(x, y) = \int_a^b K_i(x, \xi)K_j(\xi, y)d\xi.$$

---

* Our result is more general if we restrict ourselves to Volterra's method; a much more general result has been obtained by Fredholm by means of a different method. See *Acta Math.*, Vol. 27 (1903), p. 365.

† Rendiconti, *Accademie dei Lincei*, series 5, Vol. 5, 1896.

‡ The theorem can be proved with much broader restrictions on $K(x, y)$.

If we first put $i = 1$, $i + j = m$ in this formula, and then put $j = 1$, $i + j = m$, we obtain:*

$$\begin{cases} K_m(x, y) = \int_a^b K_1(x, \xi)K_{m-1}(\xi, y)d\xi, & (13a) \\ K_m(x, y) = \int_a^b K_{m-1}(x, \xi)K_1(\xi, y)d\xi. & (13b) \end{cases}$$

Volterra now proves that if the series $K_1(x, y) + \cdots + K_n(x, y) + \cdots$ *converges uniformly* in s, then the integral equation has one and only one continuous solution. We shall prove, more generally, the following theorem:

THEOREM 16: *If the series $K_1(x, y) + \cdots + K_n(x, y) + \cdots$ is uniformly evaluable in the abstract sense, then the integral equation has one and only one continuous solution.*

Since $\sum_{i=1}^{\infty} K_i(x, y)$ is evaluable,

$$-k(\xi, y) = K_1(\xi, y) + K_2(\xi, y) + \cdots + K_n(\xi, y) + \cdots,$$

and by our fundamental requirement (v), p. 2,

$$-k(\xi, y)K_1(x, \xi) = K_1(x, \xi)K_1(\xi, y) + K_1(x, \xi)K_2(\xi, y) + \cdots + K_1(x, \xi)K_n(\xi, y) + \cdots.$$

Moreover, the last series is *uniformly*† evaluable.

Hence we may integrate term by term, by Theorem 14, obtaining

$$-\int_a^b K(x, \xi)k(\xi, y)d\xi = \int_a^b K_1(x, \xi)K_1(\xi, y)d\xi + \int_a^b K_1(x, \xi)K_2(\xi, y)d\xi + \cdots + \int_a^b K_1(x, \xi)K_n(\xi, y)d\xi + \cdots$$
$$= K_2(x, y) + K_3(x, y) + \cdots + K_{n+1}(x, y) + \cdots$$

* The first of these two formulæ is the same as the definition of $K_m(x, y)$.

† The *uniform* evaluability can be established in precisely the same way as in the case of convergence.

by (13a). The series last considered has for its value, $-k(x, y) - K_1(x, y)$ so that

$$\int_a^b K(x, \xi)k(\xi, y)d\xi = K(x, y) + k(x, y).$$

By using (13b) in a similar fashion,

$$\int_a^b k(x, \xi)K(\xi, y)d\xi = K(x, y) + k(x, y).$$

The rest of the proof is the same as that given by Volterra,* who obtains as the unique continuous solution:

$$u(x) = f(x) - \int_a^b k(x, \xi)f(\xi)d\xi. \tag{14}$$

It is not difficult to construct an example for which the series $K_1(x, y) + \cdots + K_n(x, y) + \cdots$ does not converge but is, for example, Cesàro-summable of order 1. Let us look for a continuous solution of the integral equation:

$$u(x) = 1 + \frac{2}{\pi}\int_0^\pi \sin(x - y)u(y)dy.$$

Here

$$K_1(x, y) = \frac{2}{\pi}\sin(x - y), \quad K_2(x, y) = -\frac{2}{\pi}\cos(x - y),$$

$$K_3(x, y) = -\frac{2}{\pi}\sin(x - y), \quad K_4(x, y) = \frac{2}{\pi}\cos(x - y),$$

and so on, so that the series becomes

$$\begin{aligned} -k(x, y) &= \sum K_i(x, y) \\ &= \left[\frac{2}{\pi}\sin(x - y) - \frac{2}{\pi}\cos(x - y)\right](1 - 1 + 1 - 1 + \cdots), \end{aligned}$$

which is not convergent. Its summable value (C_1) is, however,

$$-k(x, y) = \frac{1}{\pi}[\sin(x - y) - \cos(x - y)]$$

* Volterra, loc. cit.

so that our solution will be:

$$u(x) = 1 + \frac{1}{\pi}\int_0^\pi [\sin(x-y) - \cos(x-y)]dy.$$

An interesting application of Cesàro-summability of order 1 has been given by L. Fejér.* It is well-known that if a function $f(x)$ satisfies Dirichlet's conditions, it may be developed into a convergent Fourier series. Fejér has shown that if $f(x)$ is finite and integrable† and of period 2π, then the Fourier development corresponding to $f(x)$ will be Cesaro-summable of order 1 to the value

$$\tfrac{1}{2}[f(x+0) + f(x-0)]$$

at all points at which the function is continuous or has a finite jump. A similar result has been obtained for the development in terms of Bessel functions by C. N. Moore.‡

We proceed to the consideration of a similar theorem in the case of the development of a function in terms of power series. If we write:

$$(15) \quad \begin{cases} s_n = f(a) + hf'(a) + \dfrac{h^2}{2!}f''(a) + \cdots + \dfrac{h^{n-1}}{(n-1)!}f^{n-1}(a), \\ R_n = f(a+h) - s_n, \end{cases}$$

then Taylor's Series with a remainder may be written

$$f(a+h) = s_n + R_n,$$

where it is found, on the assumption that $f'(x), \cdots f^{(n)}(x)$ exist, in the interval $(a, a+h)$, that §

$$(16) \qquad R_n = \frac{h^n}{n!}f^n(a+\theta h), \quad 0 < \theta < 1.$$

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* *Math. Annalen*, Bd. 58, 1904, p. 51.

† $f(x)$ may become infinite at a finite number of points.

‡ *Transactions*, Am. Math. Soc., Vol. 10 (1909), p. 391.

§ This is Lagrange's form for the remainder. See Goursat-Hedrick, loc. cit., p. 90.

From (15) it is obvious that

$$(17)\quad f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots$$

if and only if

$$\underset{n=\infty}{\mathrm{L}} R_n = 0.$$

If it should turn out that

$$\underset{n=\infty}{\mathrm{L}} R_n = k \neq 0,$$

then it follows that the series of the right member of (17) cannot represent $f(a+h)$. But if $\underset{n=\infty}{\mathrm{L}} R_n$ does not exist, though the series cannot then be convergent, it may be possible to choose a definition of sum which will give for its value $f(a+h)$. Thus we obtain from (15) and (16)

$$(18)\quad \begin{cases} \dfrac{1}{n}\sum\limits_{i=1}^{n} s_i = f(a+h) - \dfrac{1}{n}\sum\limits_{i=1}^{n} \bar{R}_i = f(a+h) - R_n, \\ \bar{R}_i = \dfrac{h^i}{i!} f^{(i)}(a+\theta_i)h, \\ R_n = \dfrac{1}{n}\sum\limits_{i=1}^{n} \bar{R}_i. \end{cases}$$

As before, we consider three possibilities. If

$$\underset{n=\infty}{\mathrm{L}} R_n = 0,$$

then

$$\underset{n=\infty}{\mathrm{L}} \frac{1}{n}\sum_{i=1}^{n} s_i = f(a+h);$$

if

$$\underset{n=\infty}{\mathrm{L}} R_n = k \neq 0, \quad \underset{n=\infty}{\mathrm{L}} \frac{1}{n}\sum_{i=1}^{n} s_i \neq f(a+h);$$

and if $\underset{n=\infty}{\mathrm{L}} R_n$ does not exist, $\underset{n=\infty}{\mathrm{L}} \frac{1}{n}\sum_{i=1}^{n} s_i$ does not exist.

This result is not satisfactory as it stands, however, because of the θ_i which appear in (18), and which may differ with i.

We shall accordingly proceed to obtain another form for R_n. We have:

$$\frac{1}{n}\sum_{i=1}^{n} s_i = \frac{nf(a) + (n-1)f'(a)\frac{h}{1!} + (n-2)f''(a)\frac{h^2}{2!} + \cdots + 2f^{(n-2)}(a)\frac{h^{n-2}}{(n-2)!} + f^{n-1}(a)\frac{h^{n-1}}{(n-1)!}}{n}. \tag{19}$$

For fixed a and h we let the difference

$$f(a+h) - \frac{1}{n}\sum_{i=1}^{n} s_i = \frac{h^p}{p}P = R_n,$$

and we consider the auxiliary function

$$\varphi(x) = \frac{1}{n}\Bigg\{ nf(a+h) - \Bigg[nf(x) + (n-1)\frac{(a+h-x)}{1!}f'(x) + (n-2)\frac{(a+h-x)^2}{2!}f''(x) + \cdots + 2\frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-2}(x) + \frac{(a+h-x)^{(n-1)}}{(n-1)!}f^{(n-1)}(x) + \frac{(a+h-x)^p}{p}nP\Bigg]\Bigg\}.$$

Since $\varphi(a) = \varphi(a+h) = 0$, it follows that $\varphi'(a+\theta h) = 0$, $0 < \theta < 1$. But

$$\begin{aligned} n\varphi'(x) = &-\Bigg[nf'(x) + (n-1)\frac{(a+h-x)}{1!}f''(x) \\ &+ (n-2)\frac{(a+h-x)^2}{2!}f'''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)\Bigg] \\ &+ \Bigg[(n-1)f'(x) + (n-2)\frac{(a+h-x)}{1!}f''(x) \\ &+ (n-3)\frac{(a+h-x)^2}{2!}f'''(x) + \cdots + (a+h-x)^{p-1}nP\Bigg] \\ = &-\Bigg[f'(x) + \frac{(a+h-x)}{1!}f''(x) + \frac{(a+h-x)^2}{2!}f'''(x) + \cdots \\ &+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - (a+h-x)^{p-1}nP\Bigg]. \end{aligned}$$

Since $\varphi'(a+\theta h) = 0$,

$$P = \frac{1}{n\lambda^{p-1}}\left\{f'(\xi) + \frac{\lambda}{1\,!}f''(\xi) + \frac{\lambda^2}{2\,!}f'''(\xi) + \cdots \right.$$
$$\left. + \frac{\lambda^{n-2}}{(n-2)\,!}f^{(n-1)}(\xi) + \frac{\lambda^{n-1}}{(n-1)\,!}f^{n}(\xi)\right\},$$

where

$$\xi = a + \theta h, \quad a + h - \xi = h(1 - \theta) = \lambda, \quad 0 < \theta < 1.$$

If we choose $p = 1$, we obtain:

$$R_n = hP = \frac{h}{n}\left\{f'(\xi) + \frac{\lambda}{1}f''(\xi) + \frac{\lambda^2}{2\,!}f'''(\xi) + \cdots \right.$$
$$\left. + \frac{\lambda^{n-2}}{(n-2)\,!}f^{n-1}(\xi) + \frac{\lambda^{n-1}}{(n-1)\,!}f^{n}(\xi)\right\}. \tag{20}$$

If now

$$\frac{1}{n}\sum_{i=1}^{n} s_i = f(a + h) - R_n,$$

then

$$\mathop{\mathrm{L}}_{n=\infty} \frac{1}{n}\sum_{i=1}^{n} s_i = f(a + h)$$

if and only if

$$\mathop{\mathrm{L}}_{n=\infty} R_n = 0.$$

We have thus proved:

THEOREM 17: *If the first n derivatives of $f(x)$ exist in the interval $(a, a + h)$, then*

$$f(a + h) = f(a) + \frac{h}{1\,!}f'(a) + \frac{h^2}{2\,!}f''(a) + \cdots + \frac{h^n}{n\,!}f^n(a) + \cdots,$$

where the infinite series is Cesàro-summable of order 1, *provided*

$$\mathop{\mathrm{L}}_{n=\infty} R_n = 0,$$

where

$$R_n = \frac{h}{n}\left\{f'(\xi) + \frac{\lambda}{1}f''(\xi) + \frac{\lambda^2}{2!}f'''(\xi) + \cdots + \frac{\lambda^{n-2}}{(n-2)!}f^{n-1}(\xi) + \frac{\lambda^{n-1}}{(n-1)!}f^n(\xi)\right\},$$

$$\xi = a + \theta h, \quad \lambda = a + h - \xi, \quad 0 < \theta < 1.$$

Turning now to the φ-definition,*

$$\varphi = \mathop{\mathrm{L}}_{n=\infty} \frac{\sum_{i=1}^{n} \varphi_i s_i}{\sum_{i=1}^{n} \varphi_i},$$

we may obtain a form for the remainder similar to (20). We shall put

$$\sum_{j=i}^{n} \varphi_j = \varphi_{in}$$

and obtain

$$\frac{\sum_{i=1}^{n} s_i\varphi_i}{\sum_{i=1}^{n} \varphi_i} = \frac{1}{\varphi_{1n}}\left[f(a)\varphi_{1n} + hf'(a)\varphi_{2n} + \frac{h^2}{2!}f''(a)\varphi_{3n} + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)\varphi_{nn}\right].$$

We now define P by the relation:

$$f(a+h) - \frac{\sum_{i=1}^{n} \varphi_i s_i}{\varphi_{1n}} = \frac{h^p}{p}P = R_n$$

* This definition is the same as that on p. 37, since

$$\mathop{\mathrm{L}}_{n=\infty} \frac{\sum_{i=1}^{n} \varphi_i}{n} = 1,$$

because

$$\mathop{\mathrm{L}}_{n=\infty} \varphi_n = 1.$$

and we construct the function:

$$\varphi(x) = \frac{1}{\varphi_{1n}}\left\{\varphi_{1n}f(a+h) - \left[\varphi_{1n}f(x) + \varphi_{2n}\frac{(a+h-x)}{1!}f'(x) + \varphi_{3n}\frac{(a+h-x)^2}{2!}f''(x) + \cdots + \varphi_{nn}\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + \frac{(a+h-x)^p}{p}\varphi_{1n}P\right]\right\},$$

Since $\varphi(a) = \varphi(a+h) = 0$, we must have $\varphi'(\xi) = 0$, $\xi = a + \theta h$, $0 < \theta < 1$. But

$$\varphi_{1n}\varphi'(x) = -\left\{\varphi_1 f'(x) + \varphi_2\frac{(a+h-x)}{1!}f''(x) + \varphi_3\frac{(a+h-x)^2}{2!}f'''(x) + \cdots + \varphi_n\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - (a+h-x)^{p-1}\varphi_{1n}P\right\}$$

so that:

$$P = \frac{1}{\varphi_{1n}}\left\{\varphi_1 f'(\xi) + \varphi_2\frac{\lambda}{1!}f''(\xi) + \varphi_3\frac{\lambda^2}{2!}f'''(\xi) + \cdots + \varphi_n\frac{\lambda^{n-1}}{(n-1)!}f^n(\xi)\right\}$$

and accordingly, if $p = 1$,

$$(21)\quad R_n = \frac{h}{\sum_{i=1}^{n}\varphi_i}\left\{\varphi_1 f'(\xi) + \varphi_2\frac{\lambda}{1!}f''(\xi) + \varphi_3\frac{\lambda^2}{2!}f'''(\xi) + \cdots + \varphi_n\frac{\lambda^{n-1}}{(n-1)!}f^n(\xi)\right\}.$$

We now turn our attention to the form of R_n for the A-definition. We set

$$\sum_{j=i}^{n} a_j(n) = a_{in}$$

and obtain:

$$\sum_{i=1}^{n} a_i(n)s_i = f(a)a_{1n} + hf'(a)a_{2n} + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)a_{nn}.$$

We define P by the relation

$$a_{1n}f(a+h) - \sum_{i=1}^{n} a_{in}s_i = hP = R_n,^*$$

and we form:

$$\varphi(x) = \left\{ a_{1n}f(a+h) - \left[a_{1n}f(x) + a_{2n}\frac{a+h-x}{1\,!}f'(x) + a_{3n}\frac{(a+h-x)^2}{2\,!}f''(x) + \cdots + a_{nn}\frac{(a+h-x)^{n-1}}{(n-1)\,!}f^{n-1}(x) + (a+h-x)P \right]\right\}.$$

Since $\varphi(a) = \varphi(a+h) = 0$, we have for $x = a + \theta h = \xi$,

$$0 = \varphi'(x) = -\left[a_1(n)f'(x) + a_2(n)\frac{a+h-x}{1\,!}f''(x) + \cdots + a_n(n)\frac{(a+h-x)^{n-1}}{(n-1)\,!}f^n(x) - P \right],$$

so that if, as before, $h(1-\theta) = \lambda$,

$$P = a_1(n)f'(\xi) + a_2(n)\frac{\lambda}{1\,!}f''(\xi) + a_3(n)\frac{\lambda^2}{2\,!}f'''(\xi) + \cdots + a_n(n)\frac{\lambda^{n-1}}{(n-1)\,!}f^n(\xi),$$

and accordingly,

$$R_n = h\left[a_1(n)f'(\xi) + a_2(n)\frac{\lambda}{1\,!}f''(\xi) + a_3(n)\frac{\lambda^2}{2\,!}f'''(\xi) + \cdots + a_n(n)\frac{\lambda^{n-1}}{(n-1)\,!}f^{(n)}(\xi) \right].$$

We may now state our result so as to include Theorem 17 as a special case.

THEOREM 18: *If the first n derivatives of $f(x)$ exist in the interval $(a, a+h)$*, then

* We previously assumed the form $(h^p/p)P$ and found $p = 1$ most convenient; we here choose $p = 1$ at the outset.

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots,$$

the infinite series being A-evaluable, provided

$$\underset{n=\infty}{\mathrm{L}} R_n = 0,$$

where

$$R_n = h\left[a_1(n)f'(\xi) + a_2(n)\frac{\lambda}{1!}f''(\xi) + \cdots + a_n(n)\frac{\lambda^{n-1}}{(n-1)!}f^n(\xi)\right]$$

$$\xi = a + \theta h, \qquad \lambda = a + h - \xi, \qquad 0 < \theta < 1.$$

We proceed now to the proof of a theorem which will again illustrate the possibility of obtaining results from very general definitions.

Any specific definition for the value of a sequence shall be briefly designated as a D-definition, if it satisfies the following requirements:

(1) the definition gives the value s whenever $\underset{n=\infty}{\mathrm{L}} s_n = s$,

(2) the definition gives ∞ whenever $\underset{n=\infty}{\mathrm{L}} s_n = \infty$.

It will be observed that every definition we have considered, either of summability or of evaluability (except* Borel's absolute summability), is a D-definition.†

It is known that if a series converges for every rearrangement of its terms, it is absolutely convergent. We now prove the following more general theorem:

THEOREM 19: *If corresponding to every arrangement (r) of the terms of a series, there exists a D-definition (D_r) which gives the series a finite value s_r, then the series converges absolutely.*

We first observe that we may assume the series to have an infinite number of terms of each sign; for otherwise, the theorem

* Here even requirement (1) is not fulfilled; see p. 56.

† We proved the satisfaction of the first requirement in all our cases except Borel's absolute summability; similar proofs can be given for the second requirement, some of which are included in Theorem 11a.

is proved, since the series cannot in that case diverge unless it diverge to infinity, which is impossible because the corresponding D-definition would give ∞, thus contradicting the hypothesis. The series has, then, an infinite number of positive terms (u_i) and an infinite number of negative terms $(-v_i)$. If each of the series

$$u_1 + u_2 + u_3 + \cdots$$

$$-v_1 - v_2 - v_3 - \cdots$$

converges, the sum converges absolutely (for we could otherwise find an arrangement r such that D_r would give ∞); and our theorem is proved. Let us assume, then, that one of the series, say the u-series, is divergent. We can accordingly choose k_1 terms from the u-series so that

$$\sum_{i=1}^{k_1} u_i > v_1 + 1,$$

then the next k_2 terms of the u-series so that

$$\sum_{i=k_1+1}^{i=k_2} u_i > v_2 + 1,$$

and so on. Now consider the arrangement

$$\sum_{i=1}^{k_1} u_i - v_1 + \sum_{i=k_1+1}^{k_2} u_i - v_2 + \cdots.$$

The sum of the first $2n$ terms is greater than n; and the sum of the first $(2n + 1)$ terms is greater than $n +$ a positive term. Hence the series diverges to ∞ for this arrangement, and accordingly the corresponding D-definition gives it the value ∞, which contradicts the hypothesis.

A series may be defined to be absolutely convergent in two ways: (1) if it converges when all its terms are made positive; (2) if it converges for every arrangement of its terms. Since the concept of absolute convergence is a useful one in the theory of convergent series, it is natural to ask whether we can introduce, correspondingly, the notion of absolute evaluability into the theory of divergent series. The two natural definitions would be: A series is absolutely evaluable if it is evaluable (1) when all its terms are made positive, (2) for every rearrangement of its terms. Consider the first definition. If the series is evaluable when all the terms are made positive, it must be convergent; for otherwise it would diverge to ∞, and could not accordingly be evaluable. As to the second definition; if a series is evaluable for every arrangement of its terms, it is, by Theorem 19, absolutely convergent. Hence neither of the definitions of absolute evaluability is useful.

§ 8. TESTS FOR CESÀRO-SUMMABILITY

As in the case of convergence, it may happen that we wish to know not what value a given series has, but whether it has any value at all. We are accordingly led to consider tests for summability.

We begin by recalling two theorems which have already been stated:

THEOREM: *A necessary condition that the series* $u_1 + u_2 + u_3 + \cdots$ *be summable* (C_r) *is*

$$\underset{n=\infty}{\mathbf{L}} \frac{u_n}{n^r} = 0.*$$

THEOREM (3): *A reducible averageable sequence with a finite number of strong limit points is Cesàro-summable of order* 1.

This is a sufficient condition for summability (C_1). We shall now consider further sufficient conditions for summability (C_1).

THEOREM 20: *If, in an alternating series, either* (*a*) *the terms decrease monotonically in absolute value, or* (*b*) *the terms increase monotonically in absolute value, while the sum of the first n terms is limited, then the series is summable* (C_1).

Let the series be $u_1+u_2+u_3+\cdots$, and $s_n=u_1+u_2+\cdots+u_n$. In case (*a*) we have $s_{2m-1} \geq s_{2m+1} \geq s_2$; $s_{2m-2} \leq s_{2m} \leq s_1$. In case (*b*) we have $s_{2m-1} \leq s_{2m+1} < A$; $s_{2m-2} \geq s_{2m} > A$. Hence in either case, $\underset{m=\infty}{\mathbf{L}}\, s_{2m+1}$ exists $= l_1$; $\underset{m=\infty}{\mathbf{L}}\, s_{2m}$ exists $= l_2$. By Theorem 3, therefore, the series is summable (C_1).

As examples, we may take:

(i) $$2 - \tfrac{3}{2} + \tfrac{4}{3} - \tfrac{5}{4} + \cdots,$$

* See p. 11.

(ii) $$1 - \tfrac{1}{2} + (\tfrac{2}{3})^2 - (\tfrac{3}{4})^3 + (\tfrac{4}{5})^4 - \cdots,$$

(iii) $$1 - 1.1 + 1.11 - 1.111 + 1.1111 - \cdots.$$

Examples (i) and (ii) illustrate case (*a*); (iii) illustrates case (*b*).

THEOREM 21: *Let*

$$s_n = \sum_{i=1}^{n} u_i, \quad S_n = \frac{1}{n}\sum_{i=1}^{n} s_i;$$

then the series $\sum_{i=1}^{\infty} u_i$ *is summable* (C_1) *if either* (*a*) $S_n \leq s_{n+1} < A$, $n \geq N$ *or* (*b*) $S_n \geq s_{n+1} > B$, $n \geq N$.

For

$$S_n - S_{n-1} = \frac{1}{n}\left[s_n - \frac{s_1 + s_2 + \cdots + s_{n-1}}{n-1}\right] = \frac{1}{n}[s_n - S_{n-1}].$$

Now by (*a*), $S_n - S_{n-1} \geq 0$, and $S_n < A$. Hence $\mathop{\mathrm{L}}_{n=\infty} S_n$ exists. Similarly for case (*b*).

THEOREM 22: *Let a series* $\sum_{i=1}^{\infty} u_i$ *satisfy the conditions*

(*a*) *the series is summable* (C_1),

(*b*) $|s_n| = |u_1 + u_2 + \cdots + u_n| < A$,

and let a set of positive constants e_i *be given such that either* (*c*) $e_i \geq e_{i+1}$ *or* (*d*) $e_i \leq e_{i+1} < A$, $i \geq N$; *then the series* $e_1u_1 + e_2u_2 + \cdots$ *is summable* (C_1).

By (*c*), $\mathop{\mathrm{L}}_{n=\infty} e_n = k$, and $e_n \geq k$.

If $k = 0$, $\sum_{i=1}^{\infty} e_iu_i$ is convergent by a well-known theorem,* and hence is summable (C_1). If $k \neq 0$, let $\delta_n = e_n - k \geq 0$. Then $\delta_n \geq \delta_{n+1} \geq 0$, and $\mathop{\mathrm{L}}_{n=\infty} \delta_n = 0$. Accordingly* the series $\sum_{i=1}^{\infty} \delta_iu_i$ is convergent, and hence summable (C_1). But $\sum_{i=1}^{\infty} ku_i$ is summable (C_1) by (*a*); so that

* See Goursat-Hedrick, *Mathematical Analysis*, p. 349, § 166.

$$\sum_{i=1}^{\infty} (\delta_i + k)u_i = \sum_{i=1}^{\infty} c_i u_i$$

is summable (C_1). Similarly for case (d).

If in the preceding theorem we put

$$\sum_{i=1}^{\infty} u_i = 1 - 1 + 1 - 1 \cdots,$$

we obtain:

Corollary 1: *If the terms of an alternating series monotonically decrease in absolute value, the series is summable* (C_1).

This is Theorem 20, case *a*.

Corollary 2: *If the terms of an alternating series remain limited, and increase monotonically in absolute value, from some point on, then the series is summable* (C_1).

Since, if $|s_n| < A$, then $|u_n| = |s_n - s_{n-1}| \leq 2A$, this corollary includes Theorem 20, case *b*, as a special case.

Before proceeding to sufficient conditions for Cesàro-summability of order higher than the first, we shall prove the following theorem,* which we shall soon need.

THEOREM 23: *If* $V = v_1 - v_2 + v_3 - v_4 + \cdots$ *is an alternating series whose terms decrease monotonically in absolute value, then the Cauchy-product of* V *by the series* $1 - 1 + 1 - 1 + \cdots$ *is summable* (C_2).

By Theorem 20, the series V is summable (C_1); hence the product is, by Theorem (J), surely summable (C_3). We wish to show that it is summable (C_2).

$$(v_1 - v_2 + v_3 - v_4 + \cdots)(1 - 1 + 1 - 1 \cdots)$$
$$= v_1 - (v_1 + v_2) + (v_1 + v_2 + v_3) - \cdots.$$

The sequence corresponding to this product series is:

(α) $\quad v_1; \quad -v_2; \quad v_1 + v_3; \quad -(v_2 + v_4); \quad (v_1 + v_3 + v_5); \quad \cdots$

* More generally, if U and V are two alternating series whose terms decrease monotonically in absolute value, then the Cauchy-product of U and V is summable (C_2). The proof is similar to that given for Theorem 24.

and the sequence for Cesàro's first mean is:

$$(\beta) \qquad v_1; \ \frac{v_1 - v_2}{2}; \ \frac{(v_1 - v_2) + (v_1 + v_3)}{3}; \ \frac{2(v_1 - v_2) + (v_3 - v_4)}{4}; \ \cdots.$$

Let us write the odd and the even elements of this sequence:

$$\begin{cases} t_{2n} = \dfrac{n(v_1 - v_2) + (n - 1)(v_3 - v_4) + \cdots + (v_{2n-1} - v_{2n})}{2n}, \\ t_{2n+1} = \dfrac{[n(v_1 - v_2) + (n - 1)(v_3 - v_4) + \cdots + (v_{2n-1} - v_{2n})] + (v_1 + v_3 + \cdots + v_{2n+1})}{2n + 1}. \end{cases}$$

Now $(v_1 - v_2) + (v_3 - v_4) + \cdots + (v_{2n-1} - v_{2n}) + \cdots$ is convergent; for if s_n denotes the sum of the first n terms of this series, we have

$$s_{n-1} < s_n < v_1, \quad \text{since} \quad v_{n+1} \leq v_n.$$

Since $\underset{n=\infty}{\mathrm{L}}\, s_n$ exists,

$$\underset{n=\infty}{\mathrm{L}} \frac{s_1 + s_2 + \cdots + s_n}{n}$$

also exists, *i. e.*,

$$\underset{n=\infty}{\mathrm{L}} \frac{n(v_1 - v_2) + (n - 1)(v_3 - v_4) + \cdots + (v_{2n-1} - v_{2n})}{n} = \underset{n=\infty}{\mathrm{L}}\, 2t_{2n}$$

exists. Furthermore, since $\underset{n=\infty}{\mathrm{L}}\, v_n$ exists (owing to the relation $0 < v_{n+1} \leq v_n$),

$$\underset{n=\infty}{\mathrm{L}}\, v_{2n+1} = l,$$

and hence

$$\underset{n=\infty}{\mathrm{L}} \frac{v_1 + v_3 + \cdots + v_{2n+1}}{n} = l.$$

Thus

$$\underset{n=\infty}{\mathrm{L}}\, t_{2n+1} = \underset{n=\infty}{\mathrm{L}}\, t_{2n} \cdot \frac{2n}{2n + 1} + \underset{n=\infty}{\mathrm{L}} \frac{v_1 + v_3 + \cdots + v_{2n+1}}{n} \cdot \frac{n}{2n + 1},$$

and each of these limits exists.

Thus by Theorem 3 the sequence β, having two and only two limits of equal weight, is summable (C_1). Hence the sequence (α) is summable (C_2); which we wished to prove.

If, in addition to the hypotheses of the preceding theorem,

$$\mathop{\mathrm{L}}_{n=\infty} v_n = 0,$$

then

$$\mathop{\mathrm{L}}_{n=\infty} \frac{v_1 + v_3 + \cdots + v_{2n+1}}{2n+1} = l = 0,$$

and

$$\mathop{\mathrm{L}}_{n=\infty} t_{2n+1} = \mathop{\mathrm{L}}_{n=\infty} t_{2n}.$$

Thus we have the theorem, due to Hardy:

THEOREM M;* *The Cauchy-product of a convergent alternating series whose terms decrease monotonically in absolute value to* 0, *by* $1 - 1 + 1 - 1 + \cdots$ *is summable* (C_1).

We now return to sufficient conditions for summability.

THEOREM 24: *Let* $u_1 - u_2 + u_3 - u_4 + \cdots$ *be an alternating series,* $u_i > 0$, *and* Δu_i† ≥ 0; *then* (*a*) *if* $\Delta^2 u_i \overline{<} 0$, *the series is summable* (C_2); *and* (*b*) *if in addition* $\mathop{\mathrm{L}}_{n=\infty} \Delta u_n = 0$, *the series is summable* (C_1).

Case (*a*). Consider the series: $u_1 - \Delta u_1 + \Delta u_2 - \Delta u_3 + \cdots$. Since $\Delta u_i > 0$, this is an alternating series, and since $\Delta^2 u_i \leq 0$, either $\Delta^2 u_i = \Delta u_{i+1} - \Delta u_i \leqq 0$, or the terms decrease monotonically. Hence by Theorem 23 the Cauchy product

$$(u_1 - \Delta u_1 + \Delta u_2 - \Delta u_3 + \cdots)(1 - 1 + 1 - 1 \cdots)$$

which is

$$= u_1 - (u_1 + \Delta u_1) + (u_1 + \Delta u_1 + \Delta u_2) - \cdots$$

$$= u_1 - u_2 + u_3 - u_4 + \cdots$$

is summable (C_2).

Case (*b*). Here the series $u_1 - \Delta u_1 + \Delta u_2 - \Delta u_3 + \cdots$

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* Bromwich, *Infinite Series,* p. 350, ex. 9. This is a special case of Theorem 27, below.

† $\Delta u_i = u_{i+1} - u_i$; $\Delta^n u_i = \Delta(\Delta^{n-1} u_i)$.

satisfies the hypothesis of Theorem M, since the terms decrease monotonically to zero. Hence the product series $u_1 - u_2 + u_3 - u_4 + \cdots$ is summable (C_1).

Thus, for example, the series

$$1 - (1 + \tfrac{1}{2}) + (1 + \tfrac{1}{2} + \tfrac{1}{3}) - \cdots$$

$$1 - \log 2 + \log 3 - \cdots$$

are summable (C_1); while the series

$$1 - 2 + 3 - 4 + \cdots$$

$$1 - \frac{2^2 + 1}{2} + \frac{3^2 + 1}{3} - \frac{4^2 + 1}{4} + \cdots$$

are summable (C_2).

THEOREM 25: *If in the series* $u_1 - u_2 + u_3 \cdots$, $u_i > 0$, $\Delta^k u_i \geq 0$,

$$\Delta^{k+1} u_i \leq 0,$$

then the series is summable (C_{k+2}); *if, in addition,*

$$\underset{n=\infty}{\mathrm{L}} \Delta^k u_n = 0,$$

then the series is summable (C_{k+1}).

Let

$$1 - 1 + 1 - \cdots = A,$$

$$d_k = \Delta^k u_1 - \Delta^k u_2 + \Delta^k u_3 - \cdots,$$

$$d_0 = u_1 - u_2 + u_3 - \cdots.$$

Then

$$\left.\begin{aligned} d_0 &= A(u_1 - d_1) \\ d_1 &= A(\Delta u_1 - d_2) \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ d_{k-1} &= A(\Delta^{k-1} u_1 - d_k) \end{aligned}\right\}.$$

Substituting the value of d_1 in the expression for d_0,

$$d_0 = Au_1 - A^2(\Delta u_1 - d_2).$$

Substituting for d_2, d_3, and so on, in turn,

$$d_0 = Au_1 - A^2\Delta u_1 + A^3\Delta^2 u_1 - \cdots \pm A^k\Delta^{k-1}u_1 \mp A^k d_k.$$

Now d_k is an alternating series whose terms decrease monotonically in absolute value. Hence d_k is summable C_1, and $A^k d_k$ is summable* (C_{k+2}). Since $d_0 \pm A^k d_k$ consists of a finite number of terms each of which is summable (C_k), or of lower order; it follows that d_0 is summable (C_{k+2}), and the first part of our theorem is proved.

If we now further assume

$$\underset{n=\infty}{\mathrm{L}}\, \Delta^k u_n = 0,$$

it is seen that d_k is convergent, and $A^k d_k$ is summable C_{k+1}. It follows, accordingly, that d_0 is summable C_{k+1}.

* It can readily be proved that A^k is summable (C_k).

The object of this section is to emphasize the value, from a practical point of view, of Theorem 11, which we restate for the sake of convenience:

THEOREM 11: *If* (1) $\underset{n=\infty}{\mathrm{L}}\, a_i(n) = 0$, *for all* i,

(2) $\underset{n=\infty}{\mathrm{L}} \sum_{i=1}^{n} a_i(n) = 1$,

(3) *either* $a_i(n) \geq 0$,

or $\sum_{i=1}^{n} |a_1(n)| < k$,*

(4) $\underset{n=\infty}{\mathrm{L}}\, s_n = s$, *or* $+\infty$,†

then

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=1}^{n} a_i(n) s_i = s,$$

or $+\infty$ *respectively.*

We have pointed out‡ that many of the definitions of summability are special cases of this theorem. But this theorem applies also to many other theorems on limits. To illustrate, we shall take some of the theorems from Bromwich's *Theory of Infinite Series.*§

THEOREM N: *If* B_n *steadily increases to* ∞, *then*

$$\underset{n=\infty}{\mathrm{L}} \frac{A_n}{B_n} = \underset{n=\infty}{\mathrm{L}} \frac{A_{n+1} - A_n}{B_{n+1} - B_n}$$

provided that the second limit exists, or is $+\infty$.

* See note (2), page 46.

† See Theorem 11*a*.

‡ See pages 43–46.

§ Pp. 377–388.

To apply Theorem 11,* we write:

$$s_1 = \frac{A_1}{B_1}; \qquad s_i = \frac{A_i - A_{i-1}}{B_i - B_{i-1}}, \quad i > 1,$$

$$a_1(n) = \frac{B_1}{B_n}; \; a_i(n) = \frac{B_i - B_{i-1}}{B_n}, \quad i > 1.$$

Since

$$\sum_{i=1}^{n} a_i(n) = 1,$$

and since it follows from the hypotheses that

$$\mathop{\mathrm{L}}_{n=\infty} a_i(n) = 0, \quad \text{and} \quad a_i(n) \geq 0,$$

we may apply Theorem 11,* and say: If

$$\mathop{\mathrm{L}}_{n=\infty} s_n = s \quad \text{or} \quad +\infty,$$

then

$$\mathop{\mathrm{L}}_{n=\infty} \sum_{i=1}^{n} a_i(n) s_i = \frac{A_1}{B_1} + \mathop{\mathrm{L}}_{n=\infty} \sum_{i=1}^{n} \frac{A_i - A_{i-1}}{B_n} = \mathop{\mathrm{L}}_{n=\infty} \frac{A_n}{B_n} = s \quad \text{or} \quad +\infty.$$

THEOREM O: *If the sequences* (s_n), (t_n) *converge to the limits* s, t, *then*

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_1 t_n + s_2 t_{n-1} + \cdots + s_n t_1}{n} = st.$$

Here choose sequence

$$s_n = s_n, \quad \text{and} \quad a_i(n) = \frac{t_{n-i+1}}{nt}.$$

Now

$$\mathop{\mathrm{L}}_{n=\infty} a_i(n) = \mathop{\mathrm{L}}_{n=\infty} \frac{1}{n} \cdot \frac{t}{t} = 0$$

and

$$\mathop{\mathrm{L}}_{n=\infty} \sum_{i=1}^{n} a_i(n) = \mathop{\mathrm{L}}_{n=\infty} \frac{1}{t} \cdot \frac{t_1 + t_2 + \cdots + t_n}{n} = 1,$$

since

$$\mathop{\mathrm{L}}_{n=\infty} t_n = t.$$

* Also Theorem 11*a*.

Furthermore,

$$\sum_{i=1}^{n} |a_i(n)| = \frac{1}{t} \frac{|t_1| + |t_2| + \cdots + |t_n|}{n} < \frac{1}{t}\frac{nk}{n} = \frac{k}{t};$$

since $|t_n| < k$, because

$$\mathop{\mathrm{L}}_{n=\infty} t_n = t$$

Hence, applying Theorem II, we obtain

$$\mathop{\mathrm{L}}_{n=\infty} \sum_{i=1}^{n} a_i(n) s_i = \mathop{\mathrm{L}}_{n=\infty} \sum_{i=1}^{n} \frac{t_{n-i+1}}{nt} \cdot s_i = \frac{1}{t} \mathop{\mathrm{L}}_{n=\infty} \frac{s_1 t_n + s_2 t_{n-1} + \cdots + s_n t_1}{n}$$

$$= s,$$

so that

$$\mathop{\mathrm{L}}_{n=\infty} \frac{s_1 t_n + s_2 t_{n-1} + \cdots + s_n t_1}{n} = s \cdot t.$$

We shall now prove Theorem L, which we stated on page 35 without proof.

THEOREM L: *If Σc_n is a divergent series of positive terms, then*

$$\mathop{\mathrm{L}}_{n=\infty} \frac{c_0 s_0 + c_1 s_1 + c_2 s_2 + \cdots + c_n s_n}{n} = \mathop{\mathrm{L}}_{n=\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n},$$

provided that the second limit exists and either (a) c_n steadily decreases, (b) c_n steadily increases, subject to the condition

$$nc_n < (c_0 + c_1 + \cdots + c_n)K,$$

where K is a fixed number.

In either case, we put

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n + 1},$$

$$a_i(n) = \frac{(i+1)(c_i - c_{i+1})}{\sum_{i=0}^{n} c_i}, \quad i \neq n,$$

$$a_n(n) = \frac{(n+1)c_n}{\sum_{i=0}^{n} c_i}.$$

Since by hypothesis

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} c_i = \infty,$$

we obtain

$$\underset{n=\infty}{\mathrm{L}}\, a_i(n) = 0.$$

Again

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(n) = \underset{n=\infty}{\mathrm{L}} \frac{(c_0-c_1)+2(c_1-c_2)+\cdots+n(c_{n-1}-c_n)+(n+1)c_n}{\sum_{i=0}^{n} c_i}$$

$$= 1.$$

Furthermore, in case (a), $a_i(n) \geq 0$, since by hypothesis $c_{n+1} \overline{<} c_n$; and in case (b),

$$\sum_{i=0}^{n} |a_i(n)| = \frac{1}{\sum_{i=0}^{n} c_i} [(c_1 - c_0) + 2(c_2 - c_1) + \cdots + n(c_n - c_{n-1}) + (n+1)c_n]$$

since by hypothesis $c_{n+1} \geq c_n$; i. e.,

$$\sum_{i=0}^{n} |a_i(n)| = \frac{1}{\sum_{i=0}^{n} c_i} [-(c_0 + c_1 + \cdots + c_{n-1}) + (2n+1)c_n]$$

$$= -1 + \frac{2(n+1)c_n}{\sum_{i=0}^{n} c_i} < 2\frac{nc_n}{\sum_{i=0}^{n} c_i}\left(\frac{n+1}{n}\right) < 4K.$$

Hence in either case (a) or (b), we have:

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(n)\sigma_i = \underset{n=\infty}{\mathrm{L}} \frac{1}{\sum_{i=0}^{n} c_i} [(c_0 - c_1)\sigma_0 + 2(c_1 - c_2)\sigma_1 + \cdots + n(c_{n-1} - c_n)\sigma_{n-1} + (n+1)c_n\sigma_n]$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{1}{\sum_{i=0}^{n} c_i} [(c_0 - c_1)s_0 + (c_1 - c_2)(s_0 + s_1) + \cdots + c_n(s_0 + s_1 + \cdots + s_n)]$$

$$= \underset{n=\infty}{\mathrm{L}} \left(\frac{\sum_{i=0}^{n} c_i s_i}{\sum_{i=0}^{n} c_i} \right) = \underset{n=\infty}{\mathrm{L}}\, \sigma_n = \underset{n=\infty}{\mathrm{L}} \frac{s_0 + s_1 + \cdots + s_n}{n}.$$

This theorem is a special case of the following more general theorem:

THEOREM P: *If Σbn, Σc_n are two divergent series of positive terms, then*

$$\underset{n=\infty}{\mathrm{L}} \frac{\sum_{i=0}^{n} c_i s_i}{\sum_{i=0}^{n} c_i} = \underset{n=\infty}{\mathrm{L}} \frac{\sum_{i=0}^{n} b_i s_i}{\sum_{i=0}^{n} b_i},$$

provided that the second limit exists and that either (a) c_n/b_n steadily decreases or (b) c_n/b_n steadily increases subject to the condition

$$\frac{c_n}{\sum_{i=0}^{n} c_i} < K \frac{b_n}{\sum_{i=0}^{n} b_i},$$

where K is fixed.

Here we put

$$\sigma_n = \frac{\sum_{i=0}^{n} b_i s_i}{\sum_{i=0}^{n} b_i}$$

so that

$$b_n s_n = (b_0 + b_1 + \cdots + b_n)\sigma_n - (b_0 + b_1 + \cdots + b_{n-1})\sigma_{n-1},$$

and set

$$a_i(n) = \left(\frac{c_i}{b_i} - \frac{c_{i+1}}{b_{i+1}}\right) \frac{b_0 + b_1 + \cdots + b_i}{c_0 + c_1 + \cdots + c_i + \cdots + c_n}, \quad i \neq n,$$

$$a_n(n) = \frac{c_n}{b_n} \frac{b_0 + b_1 + \cdots + b_n}{c_0 + c_1 + \cdots + c_n}.$$

In the first place, since

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} c_i = +\infty,$$

it follows that

$$\underset{n=\infty}{\mathrm{L}}\, a_i(n) = 0.$$

Also

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(n) = \underset{n=\infty}{\mathrm{L}} \frac{1}{\sum_{i=0}^{n} c_i} \sum_{i=0}^{n} \left(\frac{c_i}{b_i} - \frac{c_{i+1}}{b_{i+1}} \right) (b_0 + b_1 + \cdots + b_i) = 1.$$

Again in case (a), $a_i(n) \geq 0$; and in case (b),

$$\sum_{i=0}^{n} |a_i(n)| = \frac{1}{\sum_{i=0}^{n} c_i} \Big[-(c_0+c_1+\cdots+c_{n-1}) + \frac{c_n}{b_n}(2b_0+2b_1+\cdots+2b_{n-1}+b_n) \Big]$$

$$= -1 + 2\,\frac{b_0+b_1+\cdots+b_n}{n}\,c_n < 2K.$$

Now we have:

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(n)\sigma_i = \underset{n=\infty}{\mathrm{L}} \frac{\left(\frac{c_0}{b_0} - \frac{c_1}{b_1}\right) b_0 \cdot s_0 + \left(\frac{c_1}{b_1} - \frac{c_2}{b_2}\right) \times (b_0+b_1)\frac{b_0 s_0 + b_1 s_1}{b_0+b_1} + \cdots}{\sum_{i=0}^{n} c_i}$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{1}{\sum_{i=0}^{n} c_i} \Big[\left(\frac{c_0}{b_0} - \frac{c_1}{b_1}\right) b_0 s_0 + \left(\frac{c_1}{b_1} - \frac{c_2}{b_2}\right)(b_0 s_0 + b_1 s_1) + \cdots + \left(\frac{c_{n-1}}{b_{n-1}} - \frac{c_n}{b_n}\right)(b_0 s_0 + b_1 s_1 + \cdots + b_{n-1}s_{n-1}) + \frac{c_n}{b_n}(b_0 s_0 + b_1 s_1 + \cdots + b_{n-1}s_{n-1} + b_n s_n) \Big]$$

$$= \underset{n=\infty}{\mathrm{L}} \frac{1}{\sum_{i=0}^{n} c_i} [c_0 s_0 + c_1 s_1 + \cdots + c_{n-1}s_{n-1} + c_n s_n].$$

Thus in either case (a) or (b) we have the theorem established, since

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=0}^{n} a_i(n)\sigma_i = \underset{n=\infty}{\mathrm{L}} \sigma_n$$

whenever the latter exists.

§ 10. CONCLUSION

In this concluding section we propose to recall some of our main results, to show wherein they fall short of being complete, and thus to formulate the problem which remains to be solved.

Our results of § 3, concerning averageable sequences, are not of great value, since they involve a knowledge of the existence of certain limit points before the question of the existence of the averageable limit could have any significance. On the other hand, Theorem 3 is found useful in practice, in showing that certain classes of averageable sequences are summable (C_1).

Though we have discussed more general definitions, we shall confine most of our consideration in this section to the A-definition of evaluability.

It need hardly be pointed out that one of the inadequacies of the A-definition is that it may not be unique; that is, two specific sets of numbers a_{in} and b_{in}, both satisfying the conditions of the A-definition, may give different values to the same sequence. Thus the sequence $s_i = (-1)^{i+1} \log i$ has two different* values for the two different definitions:

$$a_{in} = \frac{1}{n}, \quad b_{in} = \frac{1}{n}\left[1 + (-1)^{i+1}\frac{1}{\log i}\right], \quad i > 1$$

$$= \frac{1}{n} \cdots, \quad i = 1.$$

In fact* the former definition gives the sequence (s_i) the value 0, while the latter gives it the value 1.

Two questions accordingly present themselves. First: given two A-definitions, what is a sufficient condition that one defi-

* See p. 38.

nition be a generalization* of the other? Secondly: under what conditions are the two definitions equivalent† in scope?

We shall now consider each of these questions in turn. The answer to the first question will be made clear by a few propositions.

THEOREM 26: *If*

$$\Sigma_n = a_{1n}s_1 + a_{2n}s_2 + \cdots + a_{nn}s_n$$

$$S_n = \alpha_{1n}\Sigma_1 + \alpha_{2n}\Sigma_2 + \cdots + \alpha_{nn}\Sigma_n = b_{1n}s_1 + b_{2n}s_2 + \cdots + b_{nn}s_n,$$

where a_{in} *satisfy conditions of A-evaluability,*‡

$$\sum_{i=1}^{n} b_{in} = 1, \quad \alpha_{in} \geq 0,\ddagger \quad \mathop{\mathrm{L}}_{n=\infty} \alpha_{in} = 0,$$

and if $\mathop{\mathrm{L}}_{n=\infty} \Sigma_n = s$, *then* $\mathop{\mathrm{L}}_{n=\infty} S_n = s$.

To prove this, we observe that by substituting the expression for Σ_i in the first expression given for S_n, and equating the resulting coefficients of s_i to the coefficients of s_i in the second expression for S_n, we obtain

$$a_{in}\alpha_{nn} + a_{i\ n-1}\alpha_{n-1\ n} + a_{i\ n-2}\alpha_{n-2\ n} + \cdots + a_{ii}\alpha_{in} = b_{in}.$$

Adding these equations for $i = 1, 2, \cdots n$, we get:

$$\alpha_{nn}\sum_{i=1}^{n} a_{in} + \alpha_{n-1\ n}\sum_{i=1}^{n-1} a_{i\ n-1} + \cdots + \alpha_{jn}\sum_{i=1}^{i=j} a_{ij} + \cdots + a_{1n}\cdot a_{11} = \sum_{i=1}^{n} b_{in}$$

or

$$\alpha_{nn} + \alpha_{n-1\ n} + \cdots + \alpha_{jn} + \cdots + \alpha_{1n} = 1.$$

Thus the numbers α_{in} satisfy all the conditions of Theorem 11; and our theorem is proved.

* Thus, if A_1 is (C_k) and A_2 is (C_l), then A_2 is a generalization of A_1, if $l \geq k$; i. e., if when A_1 gives to (s_n) a sum, then A_2 will give to (s_n) the same sum.

† Thus (H_r) and (C_r) are equivalent in scope; i. e., if either definition applies to s_n and gives it the sum s, then the other definition will also apply and give the sum s.

‡ See page 49, including footnote.

Now assuming $a_{nn} \neq 0$, and considering the formula

$$a_{in}\alpha_{nn} + a_{i\,n-1}\alpha_{n-1\,n} + \cdots + a_{ii}\,\alpha_{in} = b_{in}$$

as $n - i + 1$ linear equations in the $(n - i + 1)$ letters α_{in}, $\alpha_{i+1,\,n} \cdots \alpha_{nn}$; the determinant of the system of equations is

$$\begin{vmatrix} a_{nn} & 0 & \cdots & \cdots & 0 \\ a_{n-1\,n} & a_{n-1\,n-1} & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{1\,n-1} & \cdot & \cdot & a_{11} \end{vmatrix} = a_{nn}a_{n-1\,n-1} \cdots a_{11} \neq 0,$$

so that

$$\alpha_{in} = \frac{1}{a_{ii}a_{i+1\,i+1} \cdots a_{nn}} \begin{vmatrix} a_{nn} & 0 & \cdots & 0 & b_{nn} \\ a_{n-1\,n} & a_{n-1\,n-1} & \cdots & 0 & b_{n-1\,n} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{i+1\,n} & a_{i+1\,n-1} & \cdots & a_{i+1\,i+1} & b_{i+1\,n} \\ a_{in} & a_{i\,n-1} & \cdots & a_{i\,i+1} & b_{in} \end{vmatrix}$$

$$= \frac{D}{a_{ii}a_{i+1\,i+1} \cdots a_{nn}}.$$

We may then restate the previous theorem as follows:

THEOREM 27: *If a_{in}, b_{in} are numbers satisfying conditions for A-evaluability, and*

$$a_{ii} \neq 0, \quad \alpha_{in} = \frac{D}{a_{ii} \cdots a_{nn}} \geq 0,^* \quad \underset{n=\infty}{\mathrm{L}}\, \alpha_{in} = 0;$$

and if

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=1}^{n} a_{in}s_i = s,$$

then

$$\underset{n=\infty}{\mathrm{L}} \sum_{i=1}^{n} b_{in}s_i = s.$$

* See p. 49, footnote.

In particular, let a_{in} be the Cesàro coefficients for (C_r),

$$a_{in} = \frac{C_{r+n-i-1,\ n-i}}{C_{r+n-1,\ n-1}} = \frac{\dfrac{r(r+1)\cdots(r+n-i-1)}{(n-i)!}}{\dfrac{(r+1)(r+2)\cdots(r+n-1)}{(n-1)!}},$$

so that on evaluating the determinant D, we obtain

$$\alpha_{in} = \frac{1}{a_{ii}}\left(b_{in} - rb_{i+1,\ n} + \frac{r(r-1)}{1 \cdot 2} b_{i+2,\ n} \cdots + (-1)^r b_{i+r,\ n}\right),$$

or, using the notation

$$(b_{in} - b_{i+1,\ n})_r = b_{in} - rb_{i+1,\ n} + \frac{r(r-1)}{1 \cdot 2} b_{i+2,\ n} \cdots + (-1)^r b_{i+r,\ n}$$

$$\alpha_{in} = \frac{1}{a_{ii}}(b_{in} - b_{i+1,\ n})_r = \frac{1}{a_{ii}}[(b_{in} - b_{i+1,\ n})_{r-1} - (b_{i+1,\ n} - b_{i+2,\ n})_{r-1}].$$

It is evident that

$$\underset{n=\infty}{\mathrm{L}}\, \alpha_{in} = 0 \quad \text{if} \quad \underset{n=\infty}{\mathrm{L}}\, b_{in} = 0;$$

hence we may say:

THEOREM 28: *If b_{in}, corresponding to a definition B of evaluability, satisfies the condition $(b_{in} - b_{i+1,\ n})_r \geq 0$,* then if the sequence (s_n) is summable (C_r), it is also evaluable according to the B-definition.*

If we let b_{in} be the coefficients for summability (H_r), i. e.,

$$nb_{in} = (i,\ n)_{r-1} = \frac{(i,\ n)_{r-2}}{i} + \frac{(i+1,\ n)_{r-2}}{i+1} + \cdots + \frac{(n,\ n)_{r-2}}{n},$$

where

$$(i,\ n)_1 = \frac{1}{i} + \frac{1}{i+1} + \cdots + \frac{1}{n},$$

then

$$(i,\ n)_1 - (i+1,\ n)_1 = \frac{1}{i},$$

$$(i,\ n)_p - (i+1,\ n)_p = \frac{(i,\ n)_{p-1}}{i}.$$

* The condition $\sum_{i=1}^{n} |(b_{in} - b_{i+1,\ n})_r| < K$ is sufficient.

Now

$$n(b_{in} - b_{i+1,\,n})_1 = [(i,\, n)_{r-1} - (i + 1,\, n)_{r-1}] = \frac{(i,\, n)_{r-2}}{r},$$

$$n(b_{in} - b_{i+1,\,n})_2 = \frac{(i,\, n)_{r-2}}{i} - \frac{(i + 1,\, n)_{r-2}}{i + 1} = \frac{(i,\, n)_{r-2} + (i,\, n)_{r-3}}{i(i + 1)}.$$

Assume

$$n(b_{in} - b_{i+1,\,n})_j = \frac{\rho_1(i,\, n)_{r-2} + \rho_2(i,\, n)_{r-3} + \cdots + \rho_j(i,\, n)_{r-j-1}}{i(i + 1) \cdots (i + j - 1)},$$

$\rho_i > 0$.

Then

$$n(b_{in} - b_{i+1,\,n})_{j+1} = n\,[(b_{in} - b_{i+1,\,n})_j - (b_{i+1,\,n} - b_{i+2,\,n})_j]$$

$$= \frac{i\rho_1(i,\, n)_{r-2} + (\rho_1 + j\rho_2)(i,\, n)_{r-3} + \cdots + \rho_j(i,\, n)_{r-j-2}}{i(i + 1) \cdots (i + j)}$$

$$= \frac{\sigma_1(i,\, n)_{r-2} + \sigma_2(i,\, n)_{r-3} + \cdots + \sigma_{j+1}(i,\, n)_{r-j-2}}{i(i + 1) \cdots (i + j)},$$

$\sigma_i > 0$.

Hence by mathematical induction

$$n(b_{in} - b_{i+1,\,n})_j = \frac{\rho_1(i,\, n)_{r-2} + \rho_2(i,\, n)_{r-3} + \cdots + \rho_j(i,\, n)_{r-j-1}}{i(i + 1) \cdots (i + j - 1)},$$

$\rho_i > 0$, and accordingly $(b_{in} - b_{i+1,\,n})_j \geq 0$.

Thus, by our last theorem, we may say:

THEOREM Q: *If the sequence* (s_n) *is summable* (C_r), *then it is also summable* (H_r).*

The value of Theorem 27 is shown by its special cases, theorems 28 and Q. We shall give still another special case, Theorem P, due to Hardy.†

* This theorem has been proved by Ford, *Am. Journal of Math.*, Vol. 32, 1910, and by Schnee, *Math. Annalen*, Vol. 67, 1909. The converse which has been first proved by Knopp, inaugural dissertation (Berlin, 1907), can also be proved by using Theorem 29.

† *Quarterly Journal*, Vol. 38, 1907, p. 269. Hardy states that the first part of the theorem had been given by Cauchy. See p. 87 for another proof.

If $a_i > 0, \quad b_i > 0,$

$$A_n = \sum_{i=1}^{n} a_i, \quad B_n = \sum_{i=1}^{n} b_i, \quad \mathop{\mathrm{L}}_{n=\infty} B_n = \infty, \quad \mathop{\mathrm{L}}_{n=\infty} A_n = \infty,$$

and if either

$$\frac{b_i}{a_i} \geq \frac{b_{i+1}}{a_{i+1}}$$

or

$$\frac{b_i}{a_i} \leq \frac{b_{i+1}}{a_{i+1}} \quad \textit{and} \quad \frac{b_n}{B_n} < K \frac{a_n}{A_n}, \quad K > 0,$$

and if also

$$\mathop{\mathrm{L}}_{n=\infty} \frac{a_1 s_1 + \cdots + a_n s_n}{a_1 + \cdots + a_n} = s,$$

then

$$\mathop{\mathrm{L}}_{n=\infty} \frac{b_1 s_1 + \cdots + b_n s_n}{b_1 + \cdots + b_n} = s.$$

Let

$$a_{in} = \frac{a_i}{A_n}, \quad b_{in} = \frac{b_i}{B_n}$$

and

$$\alpha_{in} = \frac{1}{a_{nn} \cdots a_{ii}} \begin{vmatrix} a_n & 0 & \cdots & 0 & b_n \\ a_{n-1} & a_{n-1} & \cdots & 0 & b_{n-1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{i+1} & a_{i+1} & \cdots & a_{i+1} & b_{i+1} \\ a_i & a_i & \cdots & a_i & b_i \end{vmatrix}$$

$$= \frac{1}{B_n} \left[A_i \left(\frac{b_i}{a_i} - \frac{b_{i+1}}{a_{i+1}} \right) \right].$$

Since

$$\mathop{\mathrm{L}}_{n=\infty} B_n = \infty,$$

it follows that

$$\mathop{\mathrm{L}}_{n=\infty} \alpha_{in} = 0.$$

If further

$$\frac{b_i}{a_i} \geq \frac{b_{i+1}}{a_{i+1}},$$

then $\alpha_{in} \geq 0$. If

$$\frac{b_i}{a_i} \leq \frac{b_{i+1}}{a_{i+1}},$$

then

$$\sum_{i=1}^{n} |\alpha_{in}| = \frac{1}{B_n}\left[\left(\frac{b_2}{a_2} - \frac{b_1}{a_1}\right)A_1 + \left(\frac{b_3}{a_3} - \frac{b_2}{a_2}\right)A_2 + \cdots \right.$$
$$\left. + \left(\frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}\right)A_{n-1} + \frac{b_n}{a_n}A_n\right]$$

$$= \frac{1}{B_n}[- b_1 - b_2 \cdots - b_{n-1}] + \frac{1}{B_n}\frac{b_n}{a_n}(A_{n-1} + A_n)$$

$$= - 1 + 2\frac{b_n}{B_n}\frac{A_n}{a_n} < - 1 + 2K, \quad \text{since} \quad \frac{b_n}{B_n} < K \cdot \frac{a_n}{A_n}.$$

Thus Hardy's theorem is proved* by applying Theorem 27.†

Let us now return to the questions of page 89. The answer to the first question is found in Theorem 27, which is seen to give sufficient conditions that one of two definitions of summability be a generalization of the other. Though these sufficient conditions are fairly simple, and prove useful in leading to important theorems, it would seem extremely desirable to have sufficient conditions that D $\geq$ 0.‡

To answer the second question, we need only observe that if we can prove by Theorem 27 that definition (A) is a generalization of definition (B) and also that definition (B) is a generalization of definition (A), then (A) and (B) will be equivalent in scope.

Now let (s_n) be summable by the definition (A) and (t_n) by (B), and let one definition be a generalization of the other.

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* The proofs for this theorem, given by Hardy (loc. cit.) and by Bromwich, *Infinite Series*, p. 386, are longer.

† See p. 49, footnote 2.

‡ See p. 91 and p. 49 footnote.

Then the two sequences may be added term by term, and the resulting sequence will be summable by the more general of the two definitions. For if A is taken as the more general definition, then (s_n) is summable by (A) by hypothesis, and (t_n), being summable by (B), must also be summable by (A) which is a generalization of (B). Thus $(s_n + t_n)$ is summable by (A).

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